Tables and Goodness of Fit

How can we test if a qualitative (especially a nominal) variable appears at hypothesized frequencies?

Ex: rolling die
colors of M&M's

We can use a chi-square goodness-of-fit test.

The chi-square is yet another distribution.
It is indexed by its degrees of freedom.
For goodness-of-fit, \( df = \# \text{categories} - 1 \)

Test statistic is
\[
X^2 = \sum_{j=1}^{k} \frac{(\text{observed}_j - \text{expected}_j)^2}{\text{expected}_j}
\]

If the hypotheses are for proportions, then \( \text{expected}_j = Np_j \)
\( N = \text{total \# observations} \)
\( p_j = \text{hypothesized proportion in category } j \)

Critical values from Chi-square table A-4 with \( k-1 \) df
(always one tail, since sum of squares is positive)

Ex: You roll a die 30 times and get the following results:

is it reasonable that this die is fair?

1) Let \( p_j = \text{population probability of getting side } j \)
on a roll of a die

\( H_0: p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6} \)  fair die

\( H_1: \) not all \( p_j \)'s are \( \frac{1}{6} \)

2) \( \alpha = 0.05 \)

3) \( X^2 = \sum_{j=1}^{6} \frac{(o_j - e_j)^2}{e_j} \)

Chi-square dist'n with \( 6-1 = 5 \) df

4) \( e_j = (30)(\frac{1}{6}) = 5 \)

\( X^2 = \frac{(5-5)^2}{5} + \frac{(2-5)^2}{5} + \frac{(4-5)^2}{5} + \frac{(4-5)^2}{5} + \frac{(7-5)^2}{5} + \frac{(8-5)^2}{5} = 0.9 + 11.1 + 14.1 = 4.8 \)

Critical value = 11.071 reject \( H_0 \) if \( X^2 > 11.071 \)

5) \( 4.8 < 11.071 \) so fail to reject \( H_0 \)

6) Conclude it is reasonable the die is fair
A random sample of 55 pieces found the following colors:

\[
\begin{array}{ccccccc}
\text{Br} & \text{Y} & \text{R} & \text{O} & \text{G} & \text{B} \\
20 & 15 & 3 & 4 & 5 & 8 \\
\end{array}
\]

Are the manufacturer's claims reasonable?

1) Let \( p_j \) = population proportion of color \( j \)

\( H_0: p_1 = .3, p_2 = .2, p_3 = .2, p_4 = p_5 = p_6 = .1 \)

\( H_1: \) not all probabilities are as claimed

2) \( \alpha = .05 \)

3) \( X^2 = \sum_{j=1}^{6} \frac{(O_j - E_j)^2}{E_j} \)

4) Expected values are 55p_j \( \Rightarrow 55(0.3) = 16.5 \) \( \Rightarrow 11 \) \( \Rightarrow 5.5 \) \( \Rightarrow 5.5 \) \( \Rightarrow 5.5 \) \( \Rightarrow 5.5 \)

\( X^2 = \frac{(20-16.5)^2}{16.5} + \frac{(15-11)^2}{11} + \frac{(3-11)^2}{11} + \frac{(4-5.5)^2}{5.5} + \frac{(5-5.5)^2}{5.5} + \frac{(8-5.5)^2}{5.5} \)

\( = 9.606 \)

5) Fail to reject \( H_0 \) because 9.606 < 11.071 or 9.606 \( \neq \) 11.071

6) Conclude the manufacturer's claims are reasonable
testing for independence in tables

Recall that if two events, A and B, are independent, then \( P(A \cap B) = P(A)P(B) \).

In real life, sample proportions don't exactly match theoretical proportions because of natural variability.

How close do they need to be for independence to be reasonable?

\[
X^2 = \sum \frac{(O - E)^2}{E} \\
E = \frac{\text{row total} \times \text{column total}}{\text{grand total}}
\]

Ex: A random sample of 125 people were asked which of 3 burger chains they liked best, and the data were:

<table>
<thead>
<tr>
<th></th>
<th>Burger King</th>
<th>McDonald's</th>
<th>In-N-Out</th>
<th>Row Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>15</td>
<td>18</td>
<td>35</td>
<td>68</td>
</tr>
<tr>
<td>Female</td>
<td>35</td>
<td>12</td>
<td>10</td>
<td>57</td>
</tr>
<tr>
<td>Column Total</td>
<td>50</td>
<td>30</td>
<td>45</td>
<td>125</td>
</tr>
</tbody>
</table>

Do males have different burger preferences than females?

1) \( H_0: \) gender and burger preference are independent
2) \( H_1: \) gender and burger preference are dependent
3) \( \alpha = 0.05 \)
4) \( E = \frac{\text{row total} \times \text{column total}}{\text{grand total}} \)

\[
\begin{align*}
\text{Expected values:} & \\
M & = \frac{68 \times 125}{125} = 27.2 \\
I & = \frac{68 \times 125}{125} = 16.32 \\
F & = \frac{57 \times 125}{125} = 22.8 \\
T & = \frac{68.45 \times 125}{125} = 24.48
\end{align*}
\]

\[ \chi^2 = \sum \frac{(O - E)^2}{E} \]

\[
\begin{align*}
\chi^2 & = \frac{(15 - 27.2)^2}{27.2} + \frac{(18 - 16.32)^2}{16.32} + \frac{(35 - 24.48)^2}{24.48} + \frac{(35 - 22.8)^2}{22.8} + \frac{(12 - 13.68)^2}{13.68} + \frac{(10 - 20.52)^2}{20.52} = 22.29
\end{align*}
\]

Critical value = 5.99
5) reject \( H_0 \) because 22.29 > 5.99
6) conclude that males and females have different burger preferences or conclude that gender and burger preference are dependent.