Homework 5

Problems for Section 2.2 and 2.3

Problem 1 (10 points). Find the inverse of the following matrices, if they exist.

\[
A = \begin{bmatrix}
-2 & 3 \\
-3 & -1
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 1 & 4 \\
-1 & 1 & 2 \\
2 & 0 & 4
\end{bmatrix} \quad C = \begin{bmatrix}
-5 & 1 & -4 \\
0 & 1 & 1 \\
2 & 0 & 2
\end{bmatrix}
\]

Problem 2 (10 points). Find the value(s) of \( k \) for which the matrix \( \begin{bmatrix} k^2 & 2k \\ 8 & k \end{bmatrix} \) is not invertible.

Problem 3 (10 points). If \( A, B, \) and \( C \) are \( n \times n \) invertible matrices, does the equation

\( C^{-1}(A + X)B^{-1} = I_n \)

have a solution for \( X \)? If so, find it.

Problem 4 (10 points). Let \( D \) be a \( n \times n \) diagonal matrix, i.e.,

\[
D = \begin{bmatrix}
d_{11} & 0 & \cdots & 0 & 0 \\
0 & d_{22} & \cdots & 0 & 0 \\
0 & 0 & \cdots & d_{n-1,n-1} & 0 \\
0 & 0 & \cdots & 0 & d_{nn}
\end{bmatrix}
\]

with \( d_{ii} \neq 0 \), for all \( i = 1, 2, \cdots, n \). Find the inverse of \( D \).

Problem 5 (10 points). Let

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
-2 & -1 & 2 \\
4 & 1 & 0
\end{bmatrix}
\]

Find the second column of \( A^{-1} \) without computing the other columns.

Problem 6 (10 points). Let \( A \) be a \( n \times n \) invertible matrix. Prove that the columns \( A^T \) are linearly independent.

Problem 7 (10 points). Let \( A \) and \( B \) be \( n \times n \) matrices such that \( AB \) is invertible. Prove that both \( A \) and \( B \) are invertible.

Problem 8 (10 points). Let \( A \) be a \( n \times n \) matrix whose columns are linearly independent. Prove that the columns of \( A^2 \) are linearly independent.

Problem 9 (10 points). A square matrix \( A \) is called symmetric if \( A^T = A \). Prove that if a symmetric matrix is invertible, then its inverse is also symmetric.
Problem 10 (10 points). If $A$, $B$ and $A + B$ are all $n \times n$ invertible matrices. Prove that $A^{-1} + B^{-1}$ is invertible and the inverse is $A(A + B)^{-1}B$.

Problem 11 (50 points). Mark each statement True or False

11.1. If $A$ and $B$ are invertible, then $A + B$ is invertible.

11.2. If $A$ is $n \times n$ and not invertible, then the linear system $Ax = b$ is inconsistent.

11.3. If $(A - I)$ is invertible, then the linear system $Ax = x$ has a nonzero solution for $x$.

11.4. If a square matrix has nonzero entries on the diagonal, then $A$ is invertible.

11.5. If column of $A$ are linearly independent, then the columns of $A$ span $\mathbb{R}^n$.

11.6. Let $A$ be a square matrix. If the equation $Ax = 0$ has a nontrivial solution, then $A$ is not invertible.

11.7. A square matrix with two identical rows cannot be invertible.

11.8. A square matrix with two identical columns cannot be invertible.

11.9. A product of invertible matrices is invertible.

11.10. If $A$ and $B$ are $n \times n$ invertible matrices, then $A^{-1}B^{-1}$ is the inverse of $AB$. 