1.1 EXERCISES

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. \( x_1 + 3x_2 = 7 \)
   \(-2x_1 - 3x_2 = -5\)
   \( 3x_1 - 2x_2 = 4 \)

2. \( x_1 + 2x_2 = 4 \)
   \( 3x_1 + 4x_2 = -3 \)
   \(-2x_1 + x_2 = 10 \)

3. Find the point \((x_1, x_2, x_3)\) that lies on the line \(x_1 + 2x_2 = 4\) and on the line \(x_1 - x_2 = 1\). See the figure.

4. Find the point of intersection of the lines \(x_1 + 2x_2 = -13\)
   \(3x_1 - 2x_2 = -5\)

Consider each matrix in Exercises 5 and 6 on the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. \[
\begin{bmatrix}
1 & -4 & 3 & 7 \\
0 & 1 & 4 & 6 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & -5 & 4 & 0 \\
0 & 2 & -7 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, contain the appropriate row operations and describe the solution set of the original system.

7. \[
\begin{bmatrix}
1 & 7 & 3 & -8 \\
0 & 1 & 1 & 3 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & -5 & 4 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
1 & 3 & 0 & -2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Solve the systems in Exercises 11–14.

11. \[
\begin{bmatrix}
2 & x_1 + 3x_2 = -4 \\
x_1 + 3x_2 + 3x_3 = -2 \\
2x_1 + 2x_2 + x_3 = -2 \\
x_1 - 5x_2 + 4x_3 = -1 \\
3x_1 - 3x_2 = 8 \\
x_2 + 2x_3 + 3x_3 = 7 \\
x_1 + 2x_3 = -2 \\
x_3 - 6x_3 - 2x_3 = -4 \\
\end{bmatrix}
\]

Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

15. \[
\begin{bmatrix}
1 & -4 & 7 & 5 \\
x_1 - 4x_2 + 4x_3 = 0 \\
x_1 + 3x_2 + x_3 = 3 \\
x_2 + 3x_3 = 0 \\
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
1 & 3 & 2 & -10 \\
x_1 + x_2 + x_3 + x_4 = -1 \\
x_2 + x_3 = -1 \\
x_3 = x_4 = 5 \\
\end{bmatrix}
\]

17. Do the three lines \(2x_1 + 3x_2 = -1, 6x_1 + 5x_2 = 0, \) and \(2x_1 - 5x_2 = 7\) have a common point of intersection? Explain.

18. Do the three planes \(2x_1 + 4x_2 + 4x_3 = 4, x_2 - 2x_3 = -2, \) and \(2x_1 + 3x_2 = 0\) have at least one common point of intersection? Explain.

In Exercises 19–22, determine the value(s) of \( k \) such that the matrix is the augmented matrix of a consistent linear system.

19. \[
\begin{bmatrix}
1 & h & -3 \\
1 & 3 & h \\
1 & 4 & 2 \\
\end{bmatrix}
\]

20. \[
\begin{bmatrix}
1 & h & -5 \\
3 & 6 & 1 \\
2 & 8 & 6 \\
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
1 & 4 & -2 \\
3 & h & -6 \\
-4 & 12 & h \\
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
1 & 4 & 2 \\
3 & h & -6 \\
2 & 6 & -3 \\
\end{bmatrix}
\]

In Exercises 23 and 24, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and justify your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location (a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text.

23. Every elementary row operation is reversible.

24. A 3 x 6 matrix has six rows.

25. The solution set of a linear system involving variables \( x_1, \ldots, x_n \) is a list of numbers \( (x_1, \ldots, x_n) \) that makes each equation in the system a true statement when the values \( x_1, \ldots, x_n \) are substituted for \( x_1, \ldots, x_n \), respectively.

26. Two fundamental questions about a linear system involve existence and uniqueness.

27. Two matrices are row equivalent if they have the same number of rows.

28. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

29. Two equivalent linear systems can have different solution sets.

30. A consistent system of linear equations has one or more solutions.

31. Find an equation involving \( g, h, \) and \( k \) that makes this augmented matrix correspond to a consistent system:

\[
\begin{bmatrix}
1 & -4 & 7 & 5 \\
0 & 3 & -5 & -2 \\
0 & 5 & -9 & k \\
\end{bmatrix}
\]

32. Suppose the system below is consistent for all possible values of \( f \) and \( g \). What can you say about the coefficients \( c \) and \( d \)? Justify your answer.

\[
\begin{bmatrix}
2x_1 + 4x_2 + f = 0 \\
x_1 + dx_2 + g = 0 \\
\end{bmatrix}
\]

33. Suppose \( a, b, c, \) and \( d \) are constants such that \( a \) is not zero and the system is consistent for all possible values of \( f \) and \( g \). What can you say about the numbers \( a, b, c, \) and \( d \)? Justify your answer.

\[
\begin{bmatrix}
a_1x_1 + ax_2 = f \\
a_2x_1 + bx_2 = g \\
\end{bmatrix}
\]

34. Construct three different augmented matrices for linear systems whose solution set is \( x_1 = 3, x_2 = -2, x_3 = -1 \).

Solutions to Practice Problems

**1. a.** For “hard computation,” the best choice is to interchange equations 3 and 4. Another possibility is to multiply equation 3 by \( \frac{1}{5} \). Or, replace equation 4 by its sum with \(-\frac{1}{5}\) times row 3. (In any case, do not use the \( x_3 \) in equation 2 to eliminate the \( 4x_3 \) in equation 1. Wait until a triangular form has been reached and the \( x_3 \) terms and \( x_3 \) terms have been eliminated from the first two equations.)
during hand computations. The best strategy is to use only the reduced echelon form to solve a system! The Study Guide that accompanies this text offers several helpful suggestions for performing row operations accurately and rapidly.

**NUMERICAL NOTE**

In general, the forward phase of row reduction takes much longer than the backward phase. An algorithm for solving a system is usually measured in flops (or floating-point operations). A flop is one arithmetic operation ($+,-,\times,/$) on two real floating point numbers. For an $n \times (n + 1)$ matrix, the reduction to echelon form can take $2n^2/3 + n^2/2 - 7n/6$ flops (which is approximately $2n^2/3$ flops when $n$ is moderately large—say, $n \geq 30$). In contrast, further reduction to reduced echelon form needs at most $n^2$ flops.

**Existence and Uniqueness Questions**

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental questions posed in Section 1.1.

**EXAMPLE 5** Determine the existence and uniqueness of the solutions to the system

\[
\begin{align*}
3x_1 - 6x_2 + 6x_3 + 4x_4 &= -5 \\
3x_1 - 7x_2 + 8x_3 - 5x_4 + 3x_5 &= 9 \\
3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15
\end{align*}
\]

**SOLUTION** The augmented matrix of this system was row reduced in Example 3 to

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 2 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]  

(8)

The basic variables are $x_1, x_2,$ and $x_5$; the free variables are $x_3$ and $x_4$. There is no equation such as $0 = 1$ that would indicate an inconsistent system, so we could use back-substitution to find a solution. But the existence of a solution is already clear in (8). Also, the solution is not unique because there are free variables. Each different choice of $x_3$ and $x_4$ determines a different solution. Thus the system has infinitely many solutions.

When a system is in echelon form and contains no equation of the form $0 = b$ with $b$ nonzero, every nonzero equation contains a basic variable with a nonzero coefficient. Either the basic variables are completely determined (with no free variables) or at least one of the basic variables may be expressed in terms of one or more free variables. In the former case, there is a unique solution; in the latter case, there are infinitely many solutions (one for each choice of values for the free variables).

These remarks justify the following theorem.

**THEOREM 2** Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form

\[
\begin{bmatrix}
0 & \cdots & 0 & \ast
\end{bmatrix}
\]

with $\ast$ nonzero.

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.

**USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM**

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop, otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

**PRACTICE PROBLEMS**

1. Find the general solution of the linear system whose augmented matrix is

\[
\begin{bmatrix}
1 & -3 & -5 & 0 \\
0 & 1 & 1 & 3
\end{bmatrix}
\]

2. Find the general solution of the system

\[
\begin{align*}
x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\
-2x_1 + 4x_3 + 5x_4 - 5x_5 &= 3 \\
3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2
\end{align*}
\]

**1.2 EXERCISES**

For problems 1 and 2, determine which matrices are in reduced echelon form and which others are in echelon form.

1. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
22. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

25. Suppose a $3 \times 5$ coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

26. Suppose a $3 \times 5$ coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

27. Relate the last sentence in Theorem 2 to the concept of pivot columns: "If a linear system is consistent, then the solution is unique if and only if where the system has a unique solution."

28. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

29. A system of linear equations with fewer equations than unknowns is sometimes called an underdetermined system. Can such a system have a unique solution? Explain.

30. Give an example of an inconsistent underdetermined system of three equations in three unknowns.

31. A system of linear equations with more equations than unknowns is sometimes called an overdetermined system. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.

32. Suppose an $n \times (n + 1)$ matrix is row reduced to a reduced echelon form. Approximately what fraction of the total number of operations (flips) is involved in the backward phase of the reduction when $n = 207$ and $n = 2007$?

33. Find the interpolating polynomial $p(x) = ax + bx^2 + cx^3$ for the data $(1, 0), (2, 15), (3, 28)$. That is, find $a, b, c$ such that

$$
34. \text{[M]} \text{ A wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:} \text{ Velocity (100 lbs) \hspace{1cm} Force (100 lbs) \hspace{1cm} 0.9 \hspace{1cm} 4.6 \hspace{1cm} 8.1 \hspace{1cm} 10.15 \hspace{1cm} 30.6 \hspace{1cm} 74.3 \hspace{1cm} 119 \hspace{1cm} \text{Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile was traveling at 500 lbs. Use} p(x) = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7 + hx^8 + ix^9 + jx^{10} + kx^{11}. \text{What happens if you try to use a polynomial of degree less than 3? Try a cubic polynomial, for instance.}^3

35. \text{SOLUTIONS TO PRACTICE PROBLEMS}

1. The reduced echelon form of an augmented matrix is:

2. 3. The general solution of the system of equations is the line of intersection of the two planes.

4. True/False questions of this type will appear in many sections. Methods for verifying your answers were described before Exercises 23 and 34. Section 2.1.