Suppose the box in Fig. 3 represents some sort of electric circuit, with an input and output. Record the input voltage and current by \( V_1 \) (with voltage in volts and current in amps), and record the output voltage and current by \( V_2 \). Frequently, the transformation \( \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} \rightarrow \begin{bmatrix} v_2 \\ i_2 \end{bmatrix} \) is linear. That is, there is a matrix \( A \), called the transfer matrix, such that

\[
\begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ i_1 \end{bmatrix}
\]

**FIGURE 3** A circuit with input and output terminals.

Figure 4 shows a ladder network, where two circuits (there could be more) are connected in series, so that the output of one circuit becomes the input of the next circuit. The left circuit in Fig. 4 is called a series circuit, with resistance \( R_1 \) (in ohms).

![Diagram of a ladder network](image)

**FIGURE 4** A ladder network.

The right circuit in Fig. 4 is a shunt circuit, with resistance \( R_2 \). Using Ohm’s law and Kirchhoff’s laws, one can show that the transfer matrices of the series and shunt circuits, respectively, are

\[
\begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}
\]

**EXAMPLE 3**

a. Compute the transfer matrix of the ladder network in Fig. 4.

b. Design a ladder network whose transfer matrix is \( \begin{bmatrix} 1 & -8 \\ -5 & 5 \end{bmatrix} \).

**SOLUTION**

a. Let \( A_1 \) and \( A_2 \) be the transfer matrices of the series and shunt circuits, respectively. Then the input vector \( \mathbf{v} \) is transformed first into \( A_2 \mathbf{v} \) and then into \( A_1 (A_2 \mathbf{v}) \). The serial connection of the circuits corresponds to composition of linear transformations, so the transfer matrix of the ladder network is (note the order)

\[
A_1 A_2 = \begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -R_1 \\ 0 & 1 + R_1/R_2 \end{bmatrix}
\]

b. To factor the matrix \( \begin{bmatrix} 1 & -8 \\ -5 & 5 \end{bmatrix} \) into the product of transfer matrices, as in equation (6), look for \( R_1 \) and \( R_2 \) in Fig. 4 to satisfy

\[
\begin{bmatrix} 1 & -R_1 \\ -1/R_1 & 1 + R_1/R_2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -5 & 5 \end{bmatrix}
\]

From the (1,2)-entries, \( R_1 = 8 \) ohms, and from the (2,1)-entries, \( 1/R_1 = 5 \) ohm and \( R_2 = 1/5 = 0.2 \) ohms. With these values, the network in Fig. 4 has the desired transfer matrix.

A network transfer matrix summarizes the input–output behavior (the design specifications) of the network without reference to the interior circuits. To physically build a network with specified properties, an engineer first determines if such a network can be constructed (or realized). Then the engineer tries to factor the transfer matrix into matrices corresponding to smaller circuits that are already manufactured and ready for assembly. In the common case of alternating current, the entries in the transfer matrix are usually rational complex-valued functions. (See Exercises 19 and 20 in Section 2.4 and Example 2 in Section 3.3.) A standard problem is to find a minimal realization that uses the smallest number of electrical components.

**PRACTICE PROBLEM**

Find an LU factorization of \( A = \begin{bmatrix} 2 & 4 & -2 & 3 \\ 6 & -9 & 5 \\ -2 & -3 & 9 \\ 4 & -2 & -2 & -1 \end{bmatrix} \). [Note: It will turn out that \( A \) has only three pivot columns, so the method of Example 2 will produce only the first three columns of \( L \). The remaining two columns of \( L \) come from \( I_3 \).]

### 2.5 Exercises

Exercises 1–6, solve the equation \( Ax = b \) by using the LU factorization given for \( A \). In Exercises 1 and 2, also solve \( Ax = b \).

1. \( A = \begin{bmatrix} 3 & -7 & 2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \), \( b = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} \).

2. \( A = \begin{bmatrix} 1 & 0 & 7 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \), \( b = \begin{bmatrix} 1 \\ -7 \\ -2 \end{bmatrix} \).

3. \( A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -6 & 4 \end{bmatrix} \), \( b = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix} \).

4. \( A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \), \( b = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 4 \end{bmatrix} \).

5. \( A = \begin{bmatrix} 1 & 2 & -2 & -3 \\ 3 & -9 & 0 & 9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 \end{bmatrix} \), \( b = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \).

6. \( A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \), \( b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \).
22. (Reduced LU Factorization) With $A$ as in the Practice Problem, find a 3 x 3 matrix $R$ and a 3 x 4 matrix $C$ such that $A = RC$. Experiment with the idea of the case where $A$ is m x n, $A = LU$, and $L$ has only three nonzero rows.

23. (Rank Factorization) Suppose an m x n matrix $A$ admits a factorization $A = CD$, where $C$ is m x k and $D$ is k x n.
   a. Show that $A$ is the sum of four outer products. (See Section 2.3.4)
   b. Let $m = 400$ and $n = 100$. Explain why a computer program might prefer to store the data for $A$ in the form of two matrices $C$ and $D$.

24. (QR Factorization) Suppose $A = QR$, where $Q$ and $R$ are m x n, $R$ is invertible and upper triangular, and $Q$ has the property that $Q^T Q = I$. Show that for each $b$ in $R^n$, the equation $Ax = b$ has a unique solution. What computations with $Q$ and $R$ will produce the solution?

25. (Spectral Factorization) Suppose $A$ is a 3 x 3 matrix and $D$ is the diagonal matrix
   
   
   
   Show that this factorization is useful when computing high powers of $A$. Find fairly simple formulas for $A^2$, $A^3$, and $A^4$ (a positive integer), using $P$ and the entries in $D$.

26. (Singular Value Decomposition) Suppose $A = UDV^T$, where $U$ and $V$ are m x n matrices with the property that $U^T U = I$ and $V^T V = I$, and where $D$ is a diagonal matrix with positive numbers $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ on the diagonal. Show that $A$ is invertible, and find a formula for $A^{-1}$.

Exercises 21–26 provide a glimpse of some widely used matrix factorizations, some of which are discussed here in the text.
### 2.6 Exercises

Exercises 1–4 refer to an economy that is divided into three sectors—manufacturing, agriculture, and services. For each unit of output, manufacturing requires \(0.1\) unit from other companies in that sector, \(0.3\) unit from agriculture, and \(0.3\) unit from services. For each unit of output, agriculture uses \(0.2\) unit of its own output, \(0.8\) unit from manufacturing, and \(0.2\) unit from services. For each unit of output, the services sector consumes \(0.2\) unit from services, \(0.3\) unit from manufacturing, but no agricultural products.

1. Construct the consumption matrix for this economy, and determine what intermediate demands are created if agriculture plans to produce 100 units.

2. Determine the production levels needed to satisfy a final demand of 20 units for agriculture, with no final demand for the other sectors. (Do not compute an inverse matrix.)

3. Determine the production levels needed to satisfy a final demand of 20 units for manufacturing, with no final demand for the other sectors. (Do not compute an inverse matrix.)

4. Determine the production levels needed to satisfy a final demand of 20 units for manufacturing, 20 units for agriculture, and 6 units for services.

5. Consider the production model \(x = Cx + d\) for an economy with two sectors, where

\[
C = \begin{bmatrix}
0.5 & 0.2 \\
0.2 & 0.7 \\
\end{bmatrix}, \quad d = \begin{bmatrix}
50 \\
30 \\
\end{bmatrix}
\]

Use an inverse matrix to determine the production levels necessary to satisfy the final demand.

6. Repeat Exercise 5 with \(C = \begin{bmatrix}
2 & 0.5 \\
0.6 & 1 \\
\end{bmatrix}, d = \begin{bmatrix}
11 \\
12 \\
\end{bmatrix}.\)

7. Let \(C\) and \(d\) be as in Exercise 5.
   a. Determine the production level necessary to satisfy a final demand of 1 unit for output from sector 1.
   b. Use an inverse matrix to determine the production level necessary to satisfy a final demand of 0 unit for output from sector 1.

8. Let \(C\) be an \(n \times n\) consumption matrix whose column sums are less than 1. Let \(x\) be the production vector that satisfies a final demand \(d\), and let \(dx\) be a production vector that satisfies a different final demand \(d\).
   a. Show that if the final demand changes from \(d\) to \(d + \Delta d\), the new production level must be \(x + \Delta x\). Thus \(\Delta x\) gives the amount by which production must change in order to accommodate the change \(\Delta d\) in demand.
   b. Let \(\Delta d\) be the vector in \(\mathbb{R}^n\) with \(1\) as the first entry and 0's elsewhere. Explain why the corresponding production \(\Delta x\) is the first column of \((I - C)^{-1}\). This shows that all the first column of \((I - C)^{-1}\) gives the amounts the various sectors must produce to satisfy an increase of 1 unit in the final demand from sector 1.

9. Solve the Leontief production equation for an economy with three sectors, given that

\[
C = \begin{bmatrix}
0.3 & 0.2 & 0.2 \\
0.3 & 0.1 & 0.1 \\
0.2 & 0.1 & 0.7 \\
\end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix}
40 \\
30 \\
80 \\
\end{bmatrix}
\]

10. The consumption matrix \(C\) for the U.S. economy in 1972 has the property that every entry in the matrix \((I - C)^{-1}\) is nonnegative and positive. What does that say about the effect of raising the demand for the output of just one sector of the economy?

11. The Leontief production equation, \(x = Cx + d\), is usually accompanied by a dual price equation,

\[
p = C'p + v
\]

where \(p\) is a price vector whose entries list the price per unit for each sector's output, and \(v\) is a value added vector whose entries list the value added per unit of output. (Value added includes wages, profit, depreciation, etc.) An important fact in economics is that the gross domestic product (GDP) can be expressed in two ways:

\[\text{gross domestic product} = p'x = v'x\]

Verify the second equality. [Hint: Compute \(p'x\) in two ways.]

12. Let \(C\) be a consumption matrix such that \(C^* = 0\) as \(m \to \infty\), and for \(m = 1, 2, \ldots\) let \(D_m = I + C + \ldots + C^m\). Find a difference equation that relates \(D_m\) and \(D_{m+1}\), and thereby obtain an iterative procedure for computing \(D_m\) for \((I - C)^{-1}\).

13. [M] The consumption matrix \(C\) below is based on input-output data for the U.S. economy in 1958, with data for 81 sectors grouped into 7 larger sectors: (1) nonresidential household and personal products; (2) final metal products (such as motor vehicles); (3) basic metal products and mining; (4) basic nonmetal products and agriculture; (5) energy; (6) services; and (7) entertainment and miscellaneous products. Find the production levels needed to satisfy the final demand \(d\). (Units are in millions of dollars.)

\[
\begin{bmatrix}
0.588 & 0.064 & 0.022 & 0.004 & 0.014 & 0.008 & 0.1594 \\
0.057 & 0.265 & 0.043 & 0.009 & 0.003 & 0.001 & 0.3413 \\
0.026 & 0.1596 & 0.355 & 0.013 & 0.042 & 0.007 & 0.0286 \\
0.329 & 0.0586 & 0.2985 & 0.006 & 0.024 & 0.003 & 0.0269 \\
0.0089 & 0.001 & 0.025 & 0.0341 & 0.0237 & 0.0209 & 0.0299 \\
0.1190 & 0.091 & 0.1096 & 0.1260 & 0.1722 & 0.2368 & 0.3369 \\
0.0063 & 0.0296 & 0.0196 & 0.0089 & 0.0064 & 0.0132 & 0.0012 \\
\end{bmatrix}
\]

14. [M] The demand vector in Exercise 13 is reasonable for 1958 data, but Leontief's discussion of the economy in the reference cited there used a demand vector closer to 1964 data:

\[
d = \begin{bmatrix}
99640 & 75584 & 14444 & 33502 & 25377 & 262985 & 65268 \\
\end{bmatrix}
\]

Find the production levels needed to satisfy this demand.

15. [M] Use equation (6) to solve the problem in Exercise 13. Set \(x^{(0)} = d\), and for \(k = 1, 2, \ldots\) compute \(x^{(k)} = d - Cx^{(k-1)}\). How many steps are needed to obtain the answer in Exercise 13 for four significant figures?

1.3 Exercises

In Exercises 1 and 2, compute } u + v and } u − 2v.

1. u = \[ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \text{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \text{2. } u = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \text{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}

In Exercises 3 and 4, display the following vectors using errors xk,yk of A relative to the vector v. For each vector, show the vector, its components, and a line graph.

3. x = \[ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{4. } x = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. \[ x_1 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \quad \text{6. } x_1 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of u and v. If every vector in \( \mathbb{R}^2 \) is a linear combination of u and v, sketch the parallelogram formed by the vectors.

7. Vectors u, v, w, and \( x \)

8. Vectors u, v, w, and \( x \)

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9. \[ \begin{align*} x_1 + x_2 &= 0 \\ ax_1 + bx_2 &= c \end{align*} \quad \text{10. } \begin{align*} x_1 + x_2 &= 0 \\ ax_1 + bx_2 &= c \end{align*}

In Exercises 11 and 12, determine if \( b \) is a linear combination of \( a_1, a_2, \) and \( a_3 \).

11. \[ a_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad a_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \quad a_3 = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix} \quad \text{12. } a_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad a_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \quad a_3 = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}

In Exercises 13 and 14, determine if \( b \) is a linear combination of the vectors formed from the columns of the matrix A.

13. \[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \quad \text{14. } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 8 \end{bmatrix}

In Exercises 15 and 16, let \( a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad a_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad a_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}

15. \[ \text{Let } \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad \text{a}_1 \quad \text{a}_2 \quad \text{a}_3 \quad \text{b} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \quad \text{16. } \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad \text{a}_1 \quad \text{a}_2 \quad \text{a}_3 \quad \text{b} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}

In Exercises 17 and 18, list five vectors in Span \( \{ v_1, v_2 \} \). For each vector, show the weights on \( v_1 \) and \( v_2 \) used to generate the vector and list the three vertices of the vector. Do not make a sketch.

17. \[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{18. } v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}

In Exercises 19 and 20, give a geometric description of Span \( \{ v_1, v_2 \} \) for the vector \[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{19. } v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}

In Exercises 21 and 22, give a geometric description of Span \( \{ v_1, v_2 \} \) for the vector \[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{21. } v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}

In Exercise 23, find the standard basis vectors \( e_1, e_2, e_3 \) that span \( \mathbb{R}^3 \). Sketch the resulting vector \( v \) in \( \mathbb{R}^3 \). Determine the solution set of the linear system whose augmented matrix \[ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

In Exercises 24 and 25, use the given vector b to determine the line passing through the origin and parallel to \( v \).

24. \[ \text{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{25. } \text{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}

In Exercises 26 and 27, solve the following system of equations:

26. \[ \begin{align*} 2x_1 + 3x_2 &= 7 \\ 4x_1 + 6x_2 &= 14 \end{align*} \quad \text{27. } \begin{align*} 2x_1 + 3x_2 &= 7 \\ 4x_1 + 6x_2 &= 14 \end{align*}

In Exercises 28 and 29, solve the system of equations:

28. \[ \begin{align*} x_1 + 2x_2 &= 5 \\ 3x_1 + 6x_2 &= 15 \end{align*} \quad \text{29. } \begin{align*} x_1 + 2x_2 &= 5 \\ 3x_1 + 6x_2 &= 15 \end{align*}

In Exercises 30 and 31, solve the system of equations:

30. \[ \begin{align*} x_1 + 2x_2 &= 5 \\ 3x_1 + 6x_2 &= 15 \end{align*} \quad \text{31. } \begin{align*} x_1 + 2x_2 &= 5 \\ 3x_1 + 6x_2 &= 15 \end{align*}

The center of gravity (or center of mass) of the system is \[ \begin{bmatrix} m_1 v_1 + \cdots + m_n v_n \end{bmatrix} \]

Choose the system of point masses consisting of the following point masses (see the figure):

Point Mass
\[ v_1 = (2, -2, 4) \quad m_1 = 4 \quad 4 g \]
\[ v_2 = (1, 0, -3) \quad m_2 = 3 \quad 3 g \]
\[ v_3 = (1, -1, 2) \quad m_3 = 5 \quad 5 g \]
1.4 EXERCISES

Compute the products in Exercises 1-4 using (a) the definition, as in Example 1, and (b) the row-vector rule for computing $AX$. If the product is undefined, explain why.

1. \[ \begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \]

2. \[ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

3. \[ \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

4. \[ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} \]

In Exercises 5-8, use the definition of $AX$ to write the matrix equation as a vector equation, or vice versa.

5. \[ \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \]

6. \[ \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \]

7. \[ x_1 - x_2 + x_3 = 1 \\ \begin{array}{l} 3x_1 - 2x_2 - x_3 = -2 \\ 2x_1 + x_2 = 0 \end{array} \]

8. \[ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \]

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

9. \[ 3x_1 + 2x_2 - 3x_3 = 8 \]

10. \[ 4x_1 - x_2 = 8 \]

11. \[ \begin{bmatrix} 1 & 3 & -4 & 6 \\ 1 & 5 & 2 & -6 \\ 1 & 7 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix} \]

12. \[ \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

13. Let \( u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \) and \( A = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \). Is \( u \) in the plane \( R^2 \) spanned by the columns of \( A \)? (See the figure.) Why or why not?

14. \[ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix} \]

15. \[ \begin{bmatrix} 3 \\ -9 \\ 3 \end{bmatrix} \] and \( b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \). Show that the equation \( Ax = b \) does not have a solution for all possible \( b \), and describe the set of all \( b \) for which \( Ax = b \) does have a solution.

16. Repeat the request from Exercise 15 with \( A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \) and \( b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \).

17. Exercises 17-20 refer to the matrices \( A \) and \( B \) below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

\[ A = \begin{bmatrix} 1 & 3 & 0 & 5 \\ 1 & 4 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 4 & 2 & -8 \\ 1 & 3 & 2 & 1 \\ 0 & 2 & 9 & 5 \end{bmatrix} \]

\[ B = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 2 \\ -4 & -4 & -1 \\ 0 & 2 & 6 \\ 7 & 9 & 5 \end{bmatrix} \]

18. \[ \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \]

19. \[ \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \]

20. Let \( \alpha \) be a matrix with rows \( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \beta \) be a matrix with columns \( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \). Is \( \alpha \) in the column space of \( \beta \)? Why or why not?

21. Let \( \alpha \) be a matrix with columns \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \beta \) be a matrix with rows \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Is \( \beta \) in the column space of \( \alpha \)? Why or why not?

22. Let \( \alpha \) be a matrix with columns \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \beta \) be a matrix with rows \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Is \( \alpha \) in the row space of \( \beta \)? Why or why not?

23. Let \( \alpha \) be a matrix with columns \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \beta \) be a matrix with rows \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Is \( \beta \) in the row space of \( \alpha \)? Why or why not?

24. Every matrix equation \( Ax = b \) corresponds to a vector equation with the same solution set.

25. Any linear combination of vectors can always be written in the form \( Ax \) for a suitable matrix \( A \) and vector \( x \).

26. If the coefficient matrix \( A \) has a pivot position in every row, then the equation \( Ax = b \) is consistent.

27. The solution set of a system whose augmented matrix \( [A|b] \) is the same as the solution set of \( [A|b] \) is consistent for every \( b \) in \( \mathbb{R}^n \).

28. Every matrix equation \( Ax = b \) corresponds to a vector equation with the same solution set.

29. If the equation \( Ax = b \) is consistent, then \( b \) is in the set spanned by the columns of \( A \).

30. Any linear combination of vectors can always be written in the form \( Ax \) for a suitable matrix \( A \) and vector \( x \).

31. If the coefficient matrix \( A \) has a pivot position in every row, then the equation \( Ax = b \) is consistent.

32. The solution set of a system whose augmented matrix \( [A|b] \) is the same as the solution set of \( [A|b] \) is consistent for every \( b \) in \( \mathbb{R}^n \).

33. Every matrix equation \( Ax = b \) corresponds to a vector equation with the same solution set.

34. If the equation \( Ax = b \) is consistent, then \( b \) is in the set spanned by the columns of \( A \).

35. Any linear combination of vectors can always be written in the form \( Ax \) for a suitable matrix \( A \) and vector \( x \).

36. If the coefficient matrix \( A \) has a pivot position in every row, then the equation \( Ax = b \) is consistent.

37. The solution set of a system whose augmented matrix \( [A|b] \) is the same as the solution set of \( [A|b] \) is consistent for every \( b \) in \( \mathbb{R}^n \).
1.5 EXERCISES

In Exercises 1-4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1. \begin{align*}
2x_1 + 3x_2 + 4x_3 &= 0 \\
-x_1 + 2x_2 + 3x_3 &= 0 \\
-2x_1 + x_2 + 3x_3 &= 0
\end{align*}

2. \begin{align*}
x_1 - 2x_2 + 3x_3 &= 0 \\
-x_1 + 2x_2 - 3x_3 &= 0 \\
x_1 - 2x_2 + 3x_3 &= 0
\end{align*}

3. \begin{align*}
x_1 - 2x_2 + 3x_3 &= 0 \\
-x_1 + 2x_2 - 3x_3 &= 0 \\
x_1 - 2x_2 + 3x_3 &= 0
\end{align*}

4. \begin{align*}
x_1 - 2x_2 + 3x_3 &= 0 \\
-x_1 + 2x_2 - 3x_3 &= 0 \\
x_1 - 2x_2 + 3x_3 &= 0
\end{align*}

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

5. \begin{align*}
2x_1 + 3x_2 + 4x_3 &= 0 \\
-x_1 + 2x_2 - 3x_3 &= 0 \\
2x_1 - 2x_2 + 3x_3 &= 0
\end{align*}

6. \begin{align*}
-x_1 + 2x_2 - 3x_3 &= 0 \\
2x_1 - 2x_2 + 3x_3 &= 0 \\
-x_1 + 2x_2 - 3x_3 &= 0
\end{align*}

In Exercises 7-12, describe all solutions of \( Ax = 0 \) in parametric vector form, where \( A \) is row equivalent to the given matrix.

7. \[
\begin{bmatrix}
1 & 3 & -3 & 7 \\
0 & 1 & -4 & 5
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & -3 & -8 & 5 \\
0 & 1 & 2 & -4
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
3 & -6 & 0 & -4 \\
-2 & 4 & 2 & -4
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
-1 & -4 & 0 & -4 \\
2 & -8 & 0 & 8
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
1 & 2 & 0 & 3 & -5 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
1 & 2 & 3 & -6 & 5 \\
0 & 1 & 4 & -6 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

13. In Exercise 1, suppose the solution set of a certain system of linear equations can be described as \( x_1 = 5 + 2x_2, x_2 = -2 - 7x_3 \), with \( x_3 \) free. Use vectors to describe the solution as a line in \( \mathbb{R}^3 \).

14. Suppose the solution set of a certain system of linear equations can be described as \( x_1 = 5x_2, x_1 = 3 - 2x_3, x_2 = 2 + 5x_3 \), with \( x_2 = x_3 \) free. Use vectors to describe the solution as a line in \( \mathbb{R}^3 \).

15. Describe and compare the solution sets of \( x_1 + 3x_2 - 2x_3 = 0 \) and \( x_1 + 3x_2 - 3x_3 = -2 \).

16. Describe and compare the solution sets of \( x_1 - 2x_2 + 3x_3 = 0 \) and \( x_1 - 2x_2 + 3x_3 = 4 \).

17. In Exercise 3, follow the method of Example 3 to describe the solution of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to the solution set of Exercise 5.

18. As in Exercise 17, describe the solution of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

19. \[
\begin{bmatrix}
-2 \\
0 \\
-1
\end{bmatrix} \quad \begin{bmatrix}
-3 \\
3 \\
-2
\end{bmatrix} \quad \begin{bmatrix}
-3 \\
-2 \\
-7
\end{bmatrix}
\]

20. \[
\begin{bmatrix}
-2 \\
0 \\
-1
\end{bmatrix} \quad \begin{bmatrix}
-3 \\
3 \\
-2
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
-3 \\
0 \\
-1
\end{bmatrix} \quad \begin{bmatrix}
4 \\
0 \\
-3
\end{bmatrix} \quad \begin{bmatrix}
-3 \\
0 \\
-1
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
-3 \\
0 \\
-1
\end{bmatrix} \quad \begin{bmatrix}
4 \\
0 \\
-3
\end{bmatrix}
\]

23. a. A homogeneous equation is always consistent.

b. The equation \( Ax = 0 \) gives an explicit description of its solution set.

c. The homogeneous equation \( Ax = 0 \) has the trivial solution if and only if the equation has at least one free variable.

d. The equation \( x = p + v \) describes a line through \( p \) parallel to \( v \).

e. The solution set of \( Ax = b \) is the set of all vectors of the form \( x = p + v \), where \( v \) is any solution of the equation \( Ax = 0 \).

24. a. A homogeneous system of equations can be inconsistent.

b. If \( A \) is a non-trivial solution of \( Ax = 0 \), then every entry in \( x \) is non-zero.

c. The effect of scaling \( p \) is a vector is to move the vector in a direction parallel to \( p \).

d. The equation \( Ax = b \) is homogeneous if the zero vector is a solution.
34. Construct a 3 x 3 nonzero matrix \( A \) such that the vector \[
\begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\]
is a solution of \( Ax = 0 \).

35. Given \( A = \begin{bmatrix}
-1 & -3 \\
7 & 21 \\
-2 & -6
\end{bmatrix} \), find one nontrivial solution of \( Ax = 0 \) by inspection. \([\text{Hint: Think of the equation } Ax = 0 \text{ written as a vector equation.}]\)

36. Given \( A = \begin{bmatrix}
3 & -2 \\
6 & 4
\end{bmatrix} \), find one nontrivial solution of \( Ax = 0 \) by inspection.

37. Construct a 2 x 2 matrix \( A \) such that the solution set of the equation \( Ax = 0 \) is the line in \( \mathbb{R}^2 \) through \( (4, 1) \) and the origin. Then, find a vector \( b \) in \( \mathbb{R}^2 \) such that the solution set of \( Ax = b \) is a line parallel to the solution set of \( Ax = 0 \).

38. Let \( A \) be an \( m \times n \) matrix and let \( b \) be a vector in \( \mathbb{R}^m \) that satisfies the equation \( Ax = b \). Show that for any nonzero vector \( c \), the vector \( cAx \) also satisfies \( Ax = 0 \). \([\text{That is, show that } A(cw) = 0]\)

39. Let \( A \) be an \( m \times n \) matrix, and let \( c \) and \( d \) be vectors in \( \mathbb{R}^n \) with \( A(c + d) = 0 \). Explain why \( A(c - d) = 0 \) or \( A(c + d) = 0 \) for each pair of scalars \( c \) and \( d \).

40. Suppose \( A \) is a 2 x 3 matrix and \( b \) is a vector in \( \mathbb{R}^2 \) such that the equation \( Ax = b \) does not have a solution. Does there exist a vector \( v \) in \( \mathbb{R}^3 \) such that the equation \( A(y + v) = b + v \) has a unique solution? Discuss.

SOLUTIONS TO PRACTICE PROBLEMS

1. Row reduce the augmented matrix:

\[
\begin{bmatrix}
1 & 4 & -5 & | & 0 \\
2 & -1 & 8 & | & 9
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 4 & -5 & | & 0 \\
0 & -9 & 18 & | & 9
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 4 & -5 & | & 0 \\
0 & 1 & -2 & | & -1
\end{bmatrix}
\]

Thus \( x_1 = 4 - 3x_3, x_2 = -1 + 2x_3 \), with \( x_3 \) free. The general solution in parametric vector form is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
4 - 3x_3 \\
-1 + 2x_3 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-3 \\
2 \\
1
\end{bmatrix} + \begin{bmatrix}
4 \\
-1 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-3 \\
2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
4 \\
-1 \\
x_3
\end{bmatrix}
\]

The intersection of the two planes is the line through \( p \) in the direction of \( v \).