which can be rearranged to produce
\[
[w][v] = \frac{1}{2} \left( [w]^2 + [v]^2 \right) - [w - v]^2
\]
\[
= \frac{1}{2} \left( [v]^2 + [v]^2 \right) = v \cdot v_1 + v \cdot v_2
\]
\[
= v = v_1 + v_2
\]

The verification for \( R^2 \) is similar. When \( n > 3 \), formula (2) may be used to redefine the angle between two vectors in \( R^n \). In statistics, for instance, the value of \( \cos \theta \) defined by (2) for suitable vectors \( u \) and \( v \) is what statisticians call a correlation coefficient.

**PROBLEM PRACTICE**

Let \( a = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ c = \begin{bmatrix} 4 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \ d = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}. \)

1. Compute \( a \cdot b \) and \( \frac{a \cdot b}{a \cdot a} \).
2. Find a unit vector \( u \) in the direction of \( c \).
3. Show that \( d \) is orthogonal to \( c \).
4. Use the results of Practice Problems 2 and 3 to explain why \( d \) must be orthogonal to the unit vector \( u \).

**6.1 EXERCISES**

Compute the quantities in Exercises 1–4 using the vectors
\( u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \ x = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \ y = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 3 \end{bmatrix} \)

1. \( u \cdot v, u \cdot u, \) and \( \frac{u \cdot u}{u \cdot v} \).
2. \( u \cdot w, x \cdot w, \) and \( \frac{u \cdot w}{v \cdot w} \).
3. \( 3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \).
4. \( 4 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).
5. \( v \cdot v + w \cdot w \).
6. \( x \cdot (x \times y) \).
7. \( |v| \).
8. \( \|w\| \).

In Exercises 9–12, find a unit vector in the direction of the given vector.

9. \( \begin{bmatrix} -30 \\ 40 \end{bmatrix} \).
10. \( \begin{bmatrix} 8 \\ 3 \end{bmatrix} \).
11. \( \begin{bmatrix} 7/4 \\ 1/2 \end{bmatrix} \).
12. \( \begin{bmatrix} 8/5 \\ 2 \end{bmatrix} \).
13. Find the distance between \( x = \begin{bmatrix} -10 \\ 3 \end{bmatrix} \) and \( y = \begin{bmatrix} -5 \end{bmatrix} \).

14. Find the distance between \( u = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \) and \( z = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \).

15. Determine which pairs of vectors in Exercises 15–18 are orthogonal.
16. \( a = \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \).
17. \( a = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 6 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix} \).
18. \( a = \begin{bmatrix} 6 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix} \).

In Exercises 19 and 20, all vectors are in \( R^2 \). Mark each statement True or False. Justify each answer.

19. \( a \cdot v = v \cdot a \).
20. \( a \cdot v = a \cdot v \).

21. For any scalar \( c, \ v - c \cdot v = c \cdot v - v \).
22. If the distance from \( v \) to \( u \) equals the distance from \( u \) to \( v \), then \( u \) and \( v \) are orthogonal.
23. For a square matrix \( A \), vectors in \( \text{Col} A \) are orthogonal to vectors in \( \text{Null} A \).
24. If vectors \( v_1, \ldots, v_p \) span a subspace \( W \) and \( x \) is orthogonal to every vector in \( W \), then \( x \) is orthogonal to every vector in \( W \).

30. Let \( W \) be a subspace of \( R^n \), and let \( W' \) be the set of all vectors orthogonal to \( W \). Show that \( W' \) is a subspace of \( R^n \) following the steps.
31. Take \( x \in W' \), and let \( u \) represent any element of \( W \). Then \( x \cdot u = 0 \). Take any scalar \( c \) and show that \( c \cdot x \) is orthogonal to \( u \). (Since \( u \) was an arbitrary element of \( W \), this will show that \( W' \) is a subspace of \( R^n \).
32. Take \( u \) and \( v \) in \( W' \), and let \( a \) be any element of \( W \). Show that \( u \cdot a = -v \cdot a \). What can you conclude about \( u \cdot a \) ? Why?
33. Finish the proof that \( W' \) is a subspace of \( R^n \).
34. Show that if \( x \) is both in \( W \) and \( W' \), then \( x = 0 \).
35. [M] Construct a pair \( u, v \) of random vectors in \( R^4 \), and let

\[
A = \begin{bmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix}
\]

a. Denote the columns of \( A \) by \( a_1, \ldots, a_4 \). Compute the length of each column, and compute \( a_1 \cdot a_2, a_1 \cdot a_3, a_1 \cdot a_4, a_2 \cdot a_3, a_2 \cdot a_4, a_3 \cdot a_4 \).

b. Compute and compare the lengths of \( u, a_4, v, \) and \( u \cdot v \).

c. Use equation (2) in this section to compute the cosine of the angle between \( u \) and \( v \). Compare this with the cosine of the angle between \( a_4 \) and \( v \).

Repeat parts (b) and (c) for two other pairs of random vectors. What do you conjecture about the effect of \( A \) on vectors?

36. [M] Generate random vectors \( x, y, \) and \( x \times y \) in \( R^2 \) with integer entries (and \( y \neq 0 \)), and compute the quantities

\[
\begin{bmatrix} x \cdot y \\ x \times y \end{bmatrix}, \begin{bmatrix} x \cdot y \\ x \times y \end{bmatrix}, \begin{bmatrix} x \cdot y \\ x \times y \end{bmatrix}
\]

Repeat the computations with new random vectors \( x \) and \( y \). What do you conjecture about the mapping \( x \rightarrow T(x) = \begin{bmatrix} x \cdot y \\ x \times y \end{bmatrix} \) (for \( y \neq 0 \))? Verify your conjecture algebraically.

37. [M] Let \( A = \begin{bmatrix} 6 & -3 & -27 & -33 & -13 \\ 6 & -5 & 25 & 28 & 14 \\ 8 & -6 & 34 & 38 & 18 \end{bmatrix} \).

a. Construct a matrix \( N \) whose columns form a basis for \( \text{Null} A \), and construct a matrix \( R \) whose rows form a basis for \( \text{Row} A \) (see Section 4.6 for details). Perform a matrix computation with \( N \) and \( R \) that illustrates a fact from Theorem 3.
SOLUTION

\[ U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \quad \sqrt{3} = 1/3 \]

\[ |Ux| = \sqrt{3} + 1 + 1 = \sqrt{3} \]

\[ |x| = \sqrt{2} + 9 = \sqrt{3} \]

Theorems 6 and 7 are particularly useful when applied to square matrices. An orthogonal matrix is a square invertible matrix \( U \) such that \( U^{-1} = U^T \). By Theorem 6, such a matrix has orthonormal columns. 1 It is easy to see that any square matrix with orthonormal columns is an orthogonal matrix. Surprisingly, such a matrix must have orthonormal rows, too. See Exercises 27 and 28. Orthogonal matrices will appear frequently in Chapter 7.

EXAMPLE 7 The matrix

\[ U = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} \\ 1/\sqrt{11} & 2/\sqrt{6} \\ 1/\sqrt{11} & 1/\sqrt{6} \end{bmatrix} \quad \sqrt{66} \]

is an orthogonal matrix because it is square and because its columns are orthonormal, by Example 5. Verify that the rows are orthonormal, too!

PRACTICE PROBLEMS

1. Let \( u_1 = \begin{bmatrix} -1/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} \) and \( u_2 = \begin{bmatrix} 2/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \). Show that \( \{u_1, u_2\} \) is an orthonormal basis for \( \mathbb{R}^2 \).

2. Let \( y \) and \( L \) be as in Example 3 and Fig. 3. Compute the orthogonal projection \( \overrightarrow{y} \) of \( y \) onto \( L \). Using \( u = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \) instead of the \( u \) in Example 3.

3. Let \( U \) and \( x \) be as in Example 6, and let \( y = \begin{bmatrix} -3/2 \\ 3 \end{bmatrix} \). Verify that \( Ux \cdot y = xy \).

6.2 EXERCISES

In Exercises 1–6, determine which sets of vectors are orthogonal.

1. \( \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & 0 \end{bmatrix} \)

2. \( \begin{bmatrix} 2 & 3 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \)

3. \( \begin{bmatrix} 2 & -1 & 3 \\ -2 & -1 & 3 \\ -3 & -1 & 9 \end{bmatrix} \)

4. \( \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \)

5. \( \begin{bmatrix} 3 & -1 & 1 \\ -3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \)

6. \( \begin{bmatrix} 3 & -1 & 1 \\ -3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \)

In Exercises 7–10, show that \( \{u_1, u_2\} \) or \( \{u_1, u_2, u_3\} \) is an orthogonal basis for \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), respectively. Then express \( x \) as a linear combination of the \( u \).

7. \( u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)

8. \( u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)

9. \( u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)

10. \( u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)

b. If \( y \) is a linear combination of two vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c. If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d. A matrix with orthonormal columns is an orthogonal matrix.

e. If \( L \) and \( y \) are the orthogonal projection of \( y \) onto \( L \), then \( y \) gives the distance from \( y \) to \( L \).

24. a. Not every orthogonal set in \( \mathbb{R}^n \) is linearly independent.

b. If a set \( S = \{u_1, \ldots, u_n\} \) has the property that \( u_i \cdot u_j = 0 \) whenever \( i \neq j \), then \( S \) is an orthogonal set.

c. If the columns of an \( n \times n \) matrix \( A \) are orthonormal, then the linear mapping \( x \mapsto Ax \) preserves lengths.

d. The orthogonal projection of \( y \) onto \( v \) is the same as the orthogonal projection of \( y \) onto \( v \) when \( v \neq 0 \).

e. An orthogonal matrix is invertible.

25. Prove Theorem 7. [Hint: For \( u \), compute \( ||Ux||^2 \) and prove (b) first.]

26. Suppose \( W \) is a subspace of \( \mathbb{R}^4 \) spanned by two nonzero orthogonal vectors. Explain why \( W = \mathbb{R}^2 \).

27. Let \( U \) be a square matrix with orthogonal columns. Explain why \( U \) is invertible. ( Mention the theorems you use.)

28. Let \( U \) be an \( n \times m \) orthogonal matrix. Explain that the rows of \( U \) form an orthonormal basis of \( \mathbb{R}^m \).

29. Let \( U \) and \( V \) be \( n \times n \) orthogonal matrices. Explain why \( UU^T \) is an orthogonal matrix. That is, explain why \( UU^T \) is invertible and its inverse is \( (UU^T)^{-1} \).

30. Let \( U \) be an orthogonal matrix, and construct \( V \) by interchanging some of the columns of \( U \). Explain why \( V \) is an orthogonal matrix.

31. Show that the orthogonal projection of a vector \( y \) onto a line \( L \) through the origin in \( \mathbb{R}^3 \) does not depend on the choice of the nonzero \( u \) in \( L \) used in the formula for \( y \). To do this, suppose \( y \) and \( u \) are given and \( y \) has been computed by formula (2) in that section. Replace \( u \) in that formula by \( cu \), where \( c \) is an unspecified nonzero scalar. Show that the new formula gives the same \( y \).

32. Let \( (y_1, y_2) \) be an orthogonal set of nonzero vectors, and let \( c_1, c_2 \) be any nonzero scalars. Show that \( (c_1y_1, c_2y_2) \) is also an orthogonal set. Since orthonormality of a set is defined in terms of pairs of vectors, this shows that the vectors in an orthogonal set are normalized, the new set will still be orthogonal.

33. Given \( u \neq 0 \) in \( \mathbb{R}^3 \), let \( L = \text{Span}(u) \). Show that the mapping \( x \mapsto \text{proj}_L x \) is a linear transformation.

34. Given \( u \neq 0 \) in \( \mathbb{R}^n \), let \( L = \text{Span}(u) \). For \( y \) in \( \mathbb{R}^n \), the reflection of \( y \) in \( L \) is the point \( x \) defined by...
36. [M] In parts (a)-(f), let $U$ be the matrix formed by normalizing each column of the matrix $A$ in Exercise 35.

a. Compute $UU^T$ and $U^TU$. How do they differ?

b. Generate a random vector $y$ in $\mathbb{R}^3$, and compute $p = UU^Ty$ and $x = y - p$. Explain why $p$ is in Col $A$.

Verify that $y$ is orthogonal to $p$.

d. Verify that $Y$ is orthogonal to each column of $U$.

e. Notice that $y = p + x$, with $p$ in Col $A$. Explain why $y$ is in (Col $A)^\perp$. (The significance of this decomposition of $y$ will be explained in the next section.)

6.3 ORTHOGONAL PROJECTIONS

The orthogonal projection of a point in $\mathbb{R}^2$ onto a line through the origin has an important analogue in $\mathbb{R}^3$. Given a vector $y$ and a subspace $W$ in $\mathbb{R}^3$, there is a vector $\tilde{y}$ in $W$ such that (1) $\tilde{y}$ is the unique vector in $W$ for which $y - \tilde{y}$ is orthogonal to $W$, and (2) $\tilde{y}$ is the unique vector in $W$ closest to $y$. See Fig. 1. These two properties of $\tilde{y}$ provide the key to finding least-squares solutions of linear systems, mentioned in the introductory example for this chapter. The full story will be told in Section 6.5.

To prepare for the first theorem, observe that whenever a vector $y$ is written as a linear combination of vectors $u_1, \ldots, u_k$ in $\mathbb{R}^3$, the terms in the sum for $y$ can be grouped into two parts so that $y$ can be written as

$$y = x_1 + x_2$$

where $x_1$ is a linear combination of some of the $u_i$, and $x_2$ is a linear combination of the rest of the $u_i$. This idea is particularly useful when $\{u_1, \ldots, u_k\}$ is an orthogonal basis. Recall from Section 6.1 that $W^\perp$ denotes the set of all vectors orthogonal to a subspace $W$.

**EXAMPLE 1** Let $\{u_1, \ldots, u_k\}$ be an orthogonal basis for $\mathbb{R}^3$ and let

$$y = c_1u_1 + \cdots + c_ku_k$$

Consider the subspace $W = \operatorname{Span} \{u_1, u_2\}$, and write $y$ as the sum of a vector $x_1$ in $W$ and a vector $x_2$ in $W^\perp$.

**SOLUTION** Write

$$y = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5$$

where $x_1 = c_1u_1 + c_2u_2$ is in $\operatorname{Span} \{u_1, u_2\}$ and $x_2 = c_3u_3 + c_4u_4 + c_5u_5$ is in $\operatorname{Span} \{u_3, u_4, u_5\}$.

To show that $x_2$ is in $W^\perp$, it suffices to show that $x_2$ is orthogonal to the vectors in the basis $\{u_1, u_2\}$ for $W$. (See Section 6.1.) Using properties of the inner product, compute

$$x_2 \cdot u_1 = (c_3u_3 + c_4u_4 + c_5u_5) \cdot u_1 = c_3u_1 \cdot u_1 + c_4u_2 \cdot u_1 + c_5u_3 \cdot u_1 = 0$$

because $u_1$ is orthogonal to $u_3, u_4, u_5$. A similar calculation shows that $x_2 \cdot u_2 = 0$. Thus $x_2$ is in $W^\perp$.

The next theorem shows that the decomposition $y = x_1 + x_2$ in Example 1 can be computed without having an orthogonal basis for $\mathbb{R}^3$. It is enough to have an orthogonal basis only for $W$.
6.3 Exercises

In Exercises 1 and 2, you may assume that \( \{u_1, \ldots, u_k\} \) is an orthogonal basis for \( \mathbb{R}^k \).

1. \( u_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \) and \( W = \text{Span} \{u_1, u_3\} \).

8. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \) and \( u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \).

9. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \).

10. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \).

In Exercises 11 and 12, find the closest point to \( y \) in the subspace \( W' \) spanned by \( v_1 \) and \( v_2 \).

11. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

12. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

In Exercises 13 and 14, find the best approximation to \( x \) by a vector of the form \( c_1v_1 + c_2v_2 \).

13. \( x = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

14. \( x = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

In Exercises 15-18, let \( W \) be the subspace spanned by the \( u \)'s, and write \( y \) as the sum of a vector in \( W \) and a vector orthogonal to \( W \).

15. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \) and \( u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \).

16. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \) and \( u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \).

17. \( y = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \) and \( u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \).

In this case, \( y \) happens to be a linear combination of \( u_2 \) and \( u_3 \), so \( y \) is in \( W \). The closest point in \( W \) to \( y \) is \( y \) itself.

6.3 Orthogonal Projections

Compute

\[
\text{proj}_W y = y - \frac{y_{\cdot u_1}}{u_1 \cdot u_1} u_1 - \frac{y_{\cdot u_2}}{u_2 \cdot u_2} u_2 - \frac{y_{\cdot u_3}}{u_3 \cdot u_3} u_3
\]

where \( u_1, u_2, u_3 \) are unit vectors in \( W \), and \( y \) is a vector in \( \mathbb{R}^n \).

9. \( W \) is a subspace of \( \mathbb{R}^3 \) and \( y \) is in both \( W \) and \( W^\perp \), then \( y \) must be the zero vector.

10. In the Orthogonal Decomposition Theorem, each term in (2) for \( y \) is itself an orthogonal projection of \( y \) onto a subspace of \( W \).

11. If \( y = x_1 + x_2 \), where \( x_1 \) is in \( W \) and \( x_2 \) is in \( W^\perp \), then \( y \) must be the orthogonal projection of \( y \) onto \( W \).

12. The best approximation to \( y \) by elements of a subspace \( W \) is given by the vector \( y - \text{proj}_W y \).

13. If \( y = x \) for all \( x \) in \( \mathbb{R}^n \), then \( W = \mathbb{R}^n \).

14. Let \( W \) be a subspace of \( \mathbb{R}^n \) with an orthogonal basis \( \{w_1, \ldots, w_k\} \) and \( \{v_1, \ldots, v_l\} \) be an orthogonal basis for \( W^\perp \). Then

\[
ax = \begin{cases} \text{True} & \text{if } a \text{ is consistent} \\ \text{False} & \text{otherwise} \end{cases}
\]

If the equation \( ax = b \) is consistent, then there is a unique solution in \( W \) such that \( Ax = b \) and \( A^T Ax = A^T b \).

15. Let \( W \) be the subspace of \( \mathbb{R}^4 \) spanned by \( \{w_1, w_2, w_3, w_4\} \). Write the keystrokes or commands you use to solve this problem.

16. Let \( U \) be the matrix in Exercise 25. Find the distance from \( b \) to \( W \) in \( \mathbb{R}^4 \).

SOLUTION TO PRACTICE PROBLEM

Compute

\[
\text{proj}_W y = y - \frac{y_{\cdot u_1}}{u_1 \cdot u_1} u_1 - \frac{y_{\cdot u_2}}{u_2 \cdot u_2} u_2 - \frac{y_{\cdot u_3}}{u_3 \cdot u_3} u_3
\]

where \( u_1, u_2, u_3 \) are unit vectors in \( W \), and \( y \) is a vector in \( \mathbb{R}^n \).