INSTRUCTIONS:

1. **AMS10A**: Answer questions 1, 2 and 3 ONLY. You have 2 hours for the exam.

2. **AMS10**: Answer all questions 1 to 5. You have all 3 hours for the exam if you need them.

3. No calculators allowed (you won’t need them!)

4. Write your answers and do all working on this exam script if possible.

5. Be very clear as to how you arrived at a result (then assigning partial credit, if necessary, is easier). Mysterious answers that appear out of nowhere will not necessarily get full marks.

6. Write clearly, neatly, and in an orderly fashion. This also helps the graders give you more credit.
1. **Applications of linear systems: Traffic flow** (20 points total) Traffic flow in a small English village with four intersections is described by the diagram above. The numbers describe the flow of cars per day into and out of the intersections from outside the village, with the values of the flow between the intersections in the village unknown and denoted by $x_1, x_2, x_3, x_4$ cars per day. By noting that the flow of cars into and out of each intersection must balance, write this problem as a linear system, and (a) find the general form of the unknown flow vector $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T$.

**ANSWER (15 points):**

(b) If the flow from intersection A to intersection D is 350 cars per day (note the direction!), explain why the street from C to D is a good place to live.

**ANSWER (5 points):**
2. **Linear transformations** (20 points total)

This question concerns the following matrix representing a linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \):

\[
A = \begin{pmatrix}
2 & -3 & 1 \\
1 & 0 & 1
\end{pmatrix}
\]

(a) Find a basis for the \( \text{ker}(T) \)

**ANSWER** (8 points): 

(b) What is the dimension of the \( \text{null}(A) \)?

**ANSWER** (2 points): 

(c) Find a basis for the \( \text{ran}(T) \):

**ANSWER** (3 points): 

(question continued on next page)
(d) What is the dimension of \text{col}(A)\

\text{ANSWER (2 points):}

(e) Is the transformation \textit{onto}? (Provide a brief explanation of your answer)

\text{ANSWER (2 points):}

(f) Write out the MATLAB commands:

i. to input the matrix \textit{A}

\text{ANSWER (1 point)}

ii. a single command that would help you figure out the answer to part (a)

\text{ANSWER (1 point)}

iii. a single command that would help you figure out the answers to parts (c) and (d)

\text{ANSWER (1 point)}
3. **Determinants and inverses of matrices** (20 points total)

Consider the following $3 \times 3$ matrix

$$
A = \begin{pmatrix}
2 & -1 & 3 \\
1 & 2 & -1 \\
3 & 2 & 2
\end{pmatrix}
$$

(a) Calculate the determinant of $A$:

ANSWER (3 points):

(b) Is $A$ invertible? (Give a reason for your answer)

ANSWER (1 point):

(c) Without performing any further computation, what is the dimension of $\text{col}(A)$ and the dimension of $\text{null}(A)$?

ANSWER (2 points):

(d) Calculate $A^{-1}$, the inverse of $A$, by calculating the adjoint matrix, $\text{adj}(A)$ (do not use RREF)

ANSWER (6 points):

(e) Use the answer above to solve the matrix equation $Ax = b$ where $b = [0 \ 1 \ 1]^T$ (do not use RREF)

ANSWER (6 points):

(question continued on next page)
(f) Give single MATLAB commands to
   i. check your answer (a) above
      ANSWER (1 point):

   ii. check your answer to (d) above
      ANSWER (1 point):
4. **Orthogonality** (20 points total)

The homogeneous equation

\[ x_1 + 2x_2 - x_3 = 0 \]

defines a subspace \( U \) of \( \mathbb{R}^3 \)

(a) Write down the above linear system as a homogeneous matrix problem

\[ \text{ANSWER (2 points):} \]

(b) Show, by derivation from the matrix above, that a basis for \( U \) can be given by \( B = \{ \mathbf{v}_1, \mathbf{v}_2 \} \) where

\[ \mathbf{v}_1 = [-2 \ 1 \ 0]^T \quad \mathbf{v}_2 = [1 \ 0 \ 1]^T \]

\[ \text{ANSWER (3 points):} \]

(c) Are the basis vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) orthogonal? (Provide proof of your answer)

\[ \text{ANSWER (2 points):} \]

(question continued on next page)
(d) If the current basis vectors are not orthogonal, use projection to find \( \mathbf{v}'_1 \), the component of \( \mathbf{v}_1 \) orthogonal to (the subspace defined by) \( \mathbf{v}_2 \). That is, find \( \mathbf{v}'_1 = \text{comp}_{\mathbf{v}_2} \mathbf{v}_1 \), in order to create an orthogonal basis set of vectors \( B' = \{ \mathbf{v}'_1, \mathbf{v}_2 \} \).

Answer (5 points):

(e) Now prove that \( \mathbf{v}'_3 \), the orthogonal projection of another vector \( \mathbf{v}_3 = [1 \ 4 \ 3]^T \) onto (the subspace spanned by) \( B' \), is \( \mathbf{v}'_3 = \text{proj}_{U} \mathbf{v}_3 = [0 \ 2 \ 4]^T \).

Answer (6 points):

(f) From the above, what are the co-ordinates of the vector \( \mathbf{v}'_3 = [0 \ 2 \ 4]^T \) under the orthogonal basis \( B' \)?

Answer (2 points):
5. **Eigenvalues and eigenvectors** (20 points total)

We can consider the weather as a Markov chain process if we assume that the weather today only depends on what the weather was yesterday. Suppose that on any day, denoted by day $k$, the state of the weather is given by a vector containing the probability of it being cloudy $p_c$ and the probability of it being sunny $p_s = 1 - p_c$. That is, $W_k = [p_c, p_s]^T$. If it is a cloudy day today, the probability of it being cloudy again tomorrow is 0.4. If it is sunny today, the probability of it being sunny again tomorrow is 0.8.

(a) Write out, using a transition matrix, how the weather steps linearly from state $W_k$ to $W_{k+1}$

   **ANSWER (2 points):**

(b) Derive the characteristic equation for the eigenvalue problem involving the transition matrix

   **ANSWER (3 points):**

(c) Show that $\lambda = 1$ is an eigenvalue of this problem (i.e. satisfies the characteristic equation)

   **ANSWER (1 points):**

(d) What is the other eigenvalue of the problem?

   **ANSWER (2 points):**

(question continued on next page)
(e) In the steady state solution of the problem, what are the **probabilities** of cloudy weather and sunny weather?

ANSWER (6 points):

(f) Is the transition matrix diagonalisable? (State a reason for your answer)

ANSWER (1 point):

(g) Write down the diagonalising matrix $P$ and the diagonal matrix $D$ such that $AP = PD$ where $A$ is the transition matrix.

ANSWER (4 points):

(h) Write down MATLAB commands that would input the transition matrix, and produce the matrices $P, D$ that diagonalise $A$.

ANSWER (1 point):