1. Assume that when playing craps your point from the come-out roll is 5. Compute the probability of winning and the probability of losing the pass line bet on exactly the third roll (after the come-out roll).

**Solution:** The probability of winning exactly the third roll can be expressed as \( \Pr(A \cap A_2 \cap W) \), where \( A \) is the event that the sum in the first roll (after the come-out roll) is not 5 or 7, \( A_2 \) is the event that the sum in the second roll is not 5 or 7, and \( W \) is the event that the sum in the third roll is 5. Under the assumption of independent rolls, \( \Pr(A \cap A_2 \cap W) = \Pr(A) \times \Pr(A_2) \times \Pr(W) = \frac{26}{36} \times \frac{26}{36} \times \frac{4}{36} \). Similarly, the probability of losing on exactly the third roll is given by \( \frac{26}{36} \times \frac{26}{36} \times \frac{6}{36} \).

2. Consider the following variation of the pass line bet in the game of craps. The player rolls two (balanced) dice. If the sum on the first roll is 7 or 11, the player wins the game; if the sum on the first roll is 2, 3, or 12, the player loses the game. If the sum on the first roll is 4, 5, 6, 8, 9, or 10, then the player keeps rolling the dice until the sum is either 7 or 11, or the original value among the numbers \( \{4, 5, 6, 8, 9, 10\} \). If the original value is obtained before either 7 or 11 is obtained, then the player wins; if either 7 or 11 is obtained before the original value is obtained a second time, then the player loses. Compute the probability that the player will win the game.

**Solution:** The approach to the solution is similar to the one followed in class for the traditional version of the game of craps. Using the same notation with before, let \( p_i = \Pr(\text{sum of the first roll} = i) \), for \( i = 2, 3, \ldots, 12 \). We have \( p_2 = p_12 = 1/36, p_3 = p_11 = 2/36, p_4 = p_{10} = 3/36, p_5 = p_9 = 4/36, p_6 = p_8 = 5/36, \) and \( p_7 = 6/36 \). Therefore, \( \pi_0 = \Pr(\text{sum of first roll is 7 or 11}) = p_7 + p_{11} = 2/9 \). Moreover, let \( \pi_j = \Pr(\text{sum of the first roll is } j \text{ and the player wins at the second stage of the game}) \), for \( j = 4, 5, 6, 8, 9, 10 \). We obtain

\[
\pi_4 = \Pr(\text{sum of first roll} = 4 \text{ and then player wins}) \\
= \Pr(\text{player wins} \mid \text{sum of first roll} = 4) \times \Pr(\text{sum of first roll} = 4) \\
= \Pr(\text{sum} = 4 \mid \text{sum} = 4 \text{ or 7 or 11}) \times p_4 \\
= \left( \frac{p_4}{p_4 + p_7 + p_{11}} \right) \times p_4 = \frac{1}{44}.
\]

Similarly, \( \pi_{10} = \pi_4 = 1/44, \pi_9 = \pi_5 = 1/27, \) and \( \pi_8 = \pi_6 = 25/468 \). Finally, \( \Pr(\text{player wins the game}) = \pi_0 + \pi_4 + \pi_5 + \pi_6 + \pi_8 + \pi_9 + \pi_{10} = 0.448 \).

3. A box contains 3 coins with a head on both sides, 4 coins with a tail on both sides, and 2 fair coins. Assume that one of the 9 coins will be selected at random from the box and tossed once. Compute the probability that a head will be obtained.

**Solution:** Let \( B \) denote the event that \( H \) is obtained when the selected coin is tossed, \( A_1 \) the event that the selected coin has \( H \) on both sides, \( A_2 \) the event that the selected coin has \( T \) on both sides, and \( A_3 \) the event that the selected coin is fair. Based on the information provided, \( \Pr(A_1) = 3/9, \Pr(A_2) = 4/9, \Pr(A_3) = 2/9, \) and \( \Pr(B \mid A_1) = 1, \Pr(B \mid A_2) = 0, \Pr(B \mid A_3) = 1/2 \). Therefore, using the law of total probability, \( \Pr(B) = \Pr(A_1) \times \Pr(B \mid A_1) + \Pr(A_2) \times \Pr(B \mid A_2) + \Pr(A_3) \times \Pr(B \mid A_3) = 4/9 \).

4. Consider a box that contains five coins such that for each coin there is a different probability that a head is obtained when the coin is tossed. Assume that the probability of heads is 0 for coin 1, 1/4 for coin 2, 1/2 for coin 3, 3/4 for coin 4, and 1 for coin 5. Suppose that one of the five coins is selected at random from the box and tossed once. What is the probability that coin \( i \) (for \( i = 1, 2, 3, 4, 5 \)) was selected given that the toss resulted in a head.

**Solution:** Let \( A_i \) be the event that coin \( i \) is selected, for \( i = 1, 2, 3, 4, 5 \), and denote by \( B \) the event that \( H \) is obtained when the selected coin is tossed. Since the coin is selected at random, we have \( \Pr(A_i) = 1/5 \), for each \( i = 1, 2, 3, 4, 5 \). Moreover, \( \Pr(B \mid A_1) = 0, \Pr(B \mid A_2) = 1/4, \Pr(B \mid A_3) = 1/2, \Pr(B \mid A_4) = 3/4, \) and \( \Pr(B \mid A_5) = 1 \). Therefore, using Bayes theorem for each \( \Pr(A_i \mid B) \), we obtain \( \Pr(A_1 \mid B) = 0, \Pr(A_2 \mid B) = 0.1, \Pr(A_3 \mid B) = 0.2, \Pr(A_4 \mid B) = 0.3, \) and \( \Pr(A_5 \mid B) = 0.4 \).
5. One of three players A, B, and C will win a prize based on a draw through some random mechanism to be performed by an independent observer. Assume that each player has equal probability of winning. Suppose that the choice is made and the result is known to the observer, but not yet to the players. Player A asks the observer if he has won, but the observer refuses to tell him. Player A then asks which of players B or C has lost. The observer tells him that player B is not a winner. Player A gets excited thinking that his chance of winning has increased from 1/3 to 1/2, his argument being that since player B has lost, it is either of the other two players that will be the winner. Use conditional probabilities to show that his argument is not correct, that is,

\[ \Pr(\text{player A is the winner | observer says that player B has lost}) = \Pr(\text{player A is the winner}) = 1/3. \]

**Note:** Key to solving the problem is the following assumption on the observer’s strategy regarding his response to player’s A question (this strategy is not communicated to player A). If player A was indeed the winner, the observer would reply that either player B or player C has lost (choosing one of the two with equal probability). If player A was not the winner, he would of course reply with the name of one of the other two players that has not won.

**Solution:** Denote the event that player A, B or C is the winner by \( A, B \) or \( C \), respectively. Under the assumption that each player has equal probability of winning, we have that \( \Pr(A) = \Pr(B) = \Pr(C) = 1/3 \). Let \( R \) denote the event that the observer reveals that player B is not a winner. We need to show that \( \Pr(A \mid R) = \Pr(A) = 1/3 \). First, using the law of total probability, \( \Pr(R) = \Pr(R \mid A) \times \Pr(A) + \Pr(R \mid C) \times \Pr(C) = (1/2) \times (1/3) + (1 \times (1/3)) = 1/2 \) (note that \( \Pr(R \mid B) = 0 \), and therefore the law of total probability expression comprises only two terms in this case). Next,

\[
\Pr(A \mid R) = \frac{\Pr(R \mid A) \times \Pr(A)}{\Pr(R)} = \frac{(1/2) \times (1/3)}{(1/2)} = \frac{1}{3}
\]

Note that player A falsely interprets event \( R \) as \( B^c \) (whereas \( R \) is a strict subset of \( B^c \)), and computes \( \Pr(A \mid B^c) = \Pr(A \cap B^c)/\Pr(B^c) = \Pr(A)/(1 - \Pr(B)) = (1/3)/(2/3) = 1/2 \).

6. [Ignorant Monty] Consider the following variation of the Monty Hall problem. The player in a game show is given the choice of three doors, and will win what is behind the chosen door. Behind one door is a car, and behind each of the other two doors a goat. The car and the goats are placed at random behind the doors before the show. After the player has chosen a door, the door remains closed for the time being. The game show host, Monty Hall, opens one of the two remaining doors, but in this variation of the game, Monty does not know what is behind the doors and therefore chooses one of the two doors at random.

Assume that the player chooses door 1 and then Monty opens door 3 (relieved to find a goat). He then asks the player “Do you want to switch to door number 2?” Use Bayes theorem with appropriately adjusted probabilities to show that, in this case, there is no benefit to the player from switching.

**Solution:** Let \( C \) be the door number with the car, such that

\[ \Pr(C = 1) = \Pr(C = 2) = \Pr(C = 3) = 1/3. \]

Denote by \( H^* = (\text{Monty opens door 3 and there is a goat behind door 3}) \), which is to be distinguished from event \( H = (\text{Monty opens door 3}) \). Under the traditional version of the Monty Hall problem, events \( H \) and \( H^* \) are the same, since Monty always opens a door with a goat behind it. However, under the “ignorant Monty” variation, we need to distinguish between \( H \) and \( H^* \), since it is possible for Monty to open door 3 and find the car behind it. Now, the relevant conditional probabilities are

\[ \Pr(H^* \mid C = 1) = 1/2; \quad \Pr(H^* \mid C = 2) = 1/2; \quad \Pr(H^* \mid C = 3) = 0. \]

To formally obtain these conditional probabilities, let \( G \) be the event that door 3 has a goat behind it such that \( H^* = H \cap G \). Then, for \( i = 1, 2, 3 \),

\[ \Pr(H^* \mid C = i) = \frac{\Pr(H \cap G \cap C = i)}{\Pr(C = i)} = \frac{\Pr(H \mid G \cap C = i)\Pr(G \cap C = i)}{\Pr(C = i)} = \Pr(H \mid G \cap C = i)\Pr(G \mid C = i). \]

We have \( \Pr(G \mid C = 1) = \Pr(G \mid C = 2) = 1 \), whereas \( \Pr(G \mid C = 3) = 0 \). Therefore, \( \Pr(H^* \mid C = 3) = 0 \). Moreover, \( \Pr(H^* \mid C = 1) = \Pr(H \mid G \cap C = 1) = 1/2 \) and \( \Pr(H^* \mid C = 2) = \Pr(H \mid G \cap C = 2) = 1/2 \), since Monty will open door 3 at random.
Next, using the law of total probability, \( \Pr(H^*) = \Pr(H^* \mid C = 1)\Pr(C = 1) + \Pr(H^* \mid C = 2)\Pr(C = 2) + \Pr(H^* \mid C = 3)\Pr(C = 3) = 1/3 \). Finally, using Bayes theorem,

\[
\Pr(C = 2 \mid H^*) = \frac{\Pr(H^* \mid C = 2)\Pr(C = 2)}{\Pr(H^*)} = \frac{1}{2}; \quad \Pr(C = 1 \mid H^*) = \frac{\Pr(H^* \mid C = 1)\Pr(C = 1)}{\Pr(H^*)} = \frac{1}{2}
\]

(and \( \Pr(C = 3 \mid H^*) = 0 \)). Therefore, in this case there is no benefit from switching.

7. [Monty Hall strikes back] Consider one more variation of the Monty Hall problem. The player in a game show is given the choice of three doors, and will win what is behind the chosen door. Behind one door is a car, and behind each of the other two doors a goat. The car and the goats are placed at random behind the doors before the show. After the player has chosen a door, the door remains closed for the time being. In this variation of the game, Monty Hall (who knows what is behind each door) goes backstage and he asks the player “Do you want to switch to door number 2?” Show that, in contrast to the traditional version of the problem, there is no benefit from switching.

**Solution:** Again, let \( C \) be the door number with the car, where \( \Pr(C = 1) = \Pr(C = 2) = \Pr(C = 3) = 1/3 \). Denote by \( \{H, T\} \) the coin toss outcome, and by \( M \) the event that Monty opens door 3. Now, for any \( i = 1, 2, 3 \),

\[
\Pr(M \mid H \cap C = i) = \frac{\Pr(M \cap H \cap C = i)}{\Pr(C = i)\Pr(H)} = \frac{\Pr(M \cap H \mid C = i)\Pr(C = i)}{\Pr(H)} = 2 \times \Pr(M \cap H \mid C = i)
\]

such that \( \Pr(M \cap H \mid C = i) = (1/2) \times \Pr(M \mid H \cap C = i) \). Similarly, for \( i = 1, 2, 3 \), \( \Pr(M \cap T \mid C = i) = (1/2) \times \Pr(M \mid T \cap C = i) \). Therefore, for \( i = 1, 2, 3 \),

\[
\Pr(M \mid C = i) = \Pr(M \cap H \mid C = i) + \Pr(M \cap T \mid C = i) = ((1/2) \times \Pr(M \mid H \cap C = i)) + ((1/2) \times \Pr(M \mid T \cap C = i)).
\]

Based on the problem assumptions, we have \( \Pr(M \mid H \cap C = 1) = \Pr(M \mid T \cap C = 1) = 1/2; \Pr(M \mid H \cap C = 2) = 1; \Pr(M \mid T \cap C = 2) = 0; \) and \( \Pr(M \mid H \cap C = 3) = \Pr(M \mid T \cap C = 3) = 0 \). Substituting in the expression above for \( \Pr(M \mid C = i) \), we obtain

\[
\Pr(M \mid C = 1) = 1/2; \quad \Pr(M \mid C = 2) = 1/2; \quad \Pr(M \mid C = 3) = 0.
\]

Having obtained the conditional probabilities \( \Pr(M \mid C = i) \), for \( i = 1, 2, 3 \), we can combine with the prior probabilities, \( \Pr(C = i) \), using Bayes theorem to compute: \( \Pr(C = 1 \mid M) = \Pr(C = 2 \mid M) = 1/2. \)