AMS 80A: Gambling and Gaming (Spring 2014)

Homework 4 solutions

1. Let \( X \) be a random variable corresponding to the roll of two balanced dice, such that \( X = 2 \) if both numbers are even; \( X = 1 \) if both numbers are odd; and \( X = 0 \) if one number is odd and the other is even (the order does not matter, for example, \( X = 0 \) for outcome \((1, 2)\) as well as for outcome \((2, 1)\)). Obtain the probability distribution and the expectation of \( X \).

**Solution:**
Of the 36 outcomes in the sample space, there are 9 pairs with both numbers even, 9 pairs with both numbers odd, and 18 pairs where one number is even and the other odd. Therefore, under the assumption of equally likely outcomes in the original sample space, we have \( \Pr(X = 2) = \Pr(X = 1) = 9/36 = 1/4 \), and \( \Pr(X = 0) = 18/36 = 1/2 \). Regarding the expectation, \( E(X) = (0 \times (1/2)) + (1 \times (1/4)) + (2 \times (1/4)) = 3/4 \).

2. Let \( X \) and \( Y \) be two random variables corresponding to the roll of two balanced dice, where \( X = \) sum of the two numbers, and \( Y = \) absolute value of the difference between the two numbers. For each of \( X \) and \( Y \) obtain the probability distribution, the expectation, and the variance.

**Solution:**
The analysis for the random variable defined by the sum of the two numbers has been done in class. The probability distribution for \( X \) was found as part of homework 1, exercise 7. Moreover, \( E(X) = 7 \), and \( \text{Var}(X) = 5.833 \).

The possible values of \( Y \) are \( y = 0, 1, 2, 3, 4, 5 \). To find the corresponding probabilities, we need to count the number of outcomes among the 36 pairs of numbers that result in a difference in absolute value of 0, 1, 2, 3, 4, or 5. For instance, \( Y = 0 \) for the 6 pairs in the diagonal, \{\((1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\)\}, and thus, \( \Pr(Y = 0) = 6/36 \). Similarly, \( \Pr(Y = 1) = 10/36 \), \( \Pr(Y = 2) = 8/36 \), \( \Pr(Y = 3) = 6/36 \), \( \Pr(Y = 4) = 4/36 \), and \( \Pr(Y = 5) = 2/36 \). Applying the formulas for the expectation and variance, \( E(Y) = 1.94 \), and \( \text{Var}(Y) = 2.0525 \).

3. Exercises 2 and 8 from Chapter 2 of the Rodriguez and Mendes draft book.

**Solution:**
Exercise 2: The expected profit is given by \((((-1) \times 1/2)) + (0.25 \times 1/2)) = -$0.375 \) for wager 1, by \(((1) \times 1/3)) + (0 \times 1/3)) + (0.5 \times 1/3)) = -$0.167 \) for wager 2, and by \(((0) \times 1/2)) + (0 \times 1/6)) + (0.5 \times 1/3)) = -$0.333 \) for wager 3. Therefore, a “rational player” would select wager 2.

Exercise 8: The expected profit is given by \((100 \times 0.8) + (150 \times 0.1) + (200 \times 0.05) + (500 \times 0.05) = $50 \) for stock A, by \((65 \times 0.8) + (15 \times 0.1) + (40 \times 0.05) + (50 \times 0.05) = $50 \) for stock B, and by \((100 \times 0.5) + (150 \times 0.2) + (200 \times 0.15) + (500 \times 0.15) = $25 \) for stock C. Therefore, stock C is eliminated as a choice based on expected profit. Computing the standard deviation of the profit for stocks A and B, we obtain $149.16 and $33.28, respectively, resulting in stock B as the “rational player” choice.

4. Consider an (unbiased) American roulette wheel, which has 38 pockets. Two pockets have green color and are marked by 0 and 00. The remaining 36 pockets are numbered from 1 to 36 and alternate in color between red and black (such that overall there are 18 red, 18 black, and 2 green pockets). Consider the “top line” bet for which the winning pockets are \{0, 00, 1, 2, 3\}. The payout for this bet is 6 to 1; that is, if the player bets a unit of money and the wheel spin results in 0, 00, 1, 2, or 3, then the player keeps the unit and receives an additional 6 units; if the ball falls into one of the other pockets, the player loses the unit of money. If you place a $10 “top line” bet, what is your expected profit and what is the standard deviation of the profit?

**Solution:**
The profit from this bet takes values $60 and -$10 with respective probabilities 5/38 and 33/38. Therefore, the expected profit is \((60 \times 5/38) + ((-10) \times 33/38)) = -$0.7895 \). Moreover, the variance of the profit is given by \(((60 + 0.7895)^2 \times 5/38) + ((-10 + 0.7895)^2 \times 33/38)) = 559.903 \), and the standard deviation of the profit by \(\sqrt{559.903} = $23.66\).
5. A game/bet is *fair* if the corresponding expected profit is 0, that is, on average players neither win nor lose. Consider again the “top line” bet in American roulette from the previous exercise. How should the payout be modified such that the “top line” bet becomes fair?

**Solution:**
Let $z$ denote the payout for an $1 “top line” bet that corresponds to a fair game. Then, setting the expected profit equal to 0, we get $(z \times (5/38)) + ((-1) \times (33/38)) = 0$, which when solved for $z$ yields $z = 6.6$. Therefore, the payout for a fair “top line” bet is 6.6 to 1.

6. The game of chance *chuck-a-luck* is based on the outcome of the roll of three dice (assume balanced dice). The standard bet for this game is the “numbers bet” where the player bets on one of the numbers \{1, 2, 3, 4, 5, 6\}. The player loses if the number bet on does not appear on any of the three dice. The player wins if the number bet on appears on at least one of the dice. The corresponding payout is: 1 to 1, if the number appears on exactly one die; 2 to 1, if the number appears on exactly two dice; 10 to 1, if the number appears on all three dice.

Assume that you bet $1 on number 6, and therefore, under the rules of the game above, you lose $1 if 6 does not appear on any of the dice, you win $1 if 6 appears on one die, you win $2 if 6 appears on two dice, and you win $10 if the outcome is (6,6,6). What is your expected profit? Compute also the standard deviation of the profit under this bet.

**Solution:**
The profit from this bet takes values: $10 with probability 1/216; $2 with probability 15/216; $1 with probability 75/216; and $−1 with probability 125/216. The probabilities are obtained under the assumption of equally likely outcomes for the experiment involving the roll of three balanced dice, which has a total number of $6^3 = 216$ outcomes. For instance, winning $2 corresponds to outcomes of the form (6, a, b) or (a, 6, b) or (b, 6, a), where a = 1, 2, 3, 4, 5; therefore, 15 out of the 216 outcomes result in winning $2. Applying the expectation and variance formulas, we obtain $E(\text{profit}) = −0.0463$ and $\text{Var}(\text{profit}) = 1.665$, which yields $SD(\text{profit}) = \sqrt{1.665} = $1.29.