1. The martingale betting system is one of the several betting strategies that have been devised for the game of roulette. Under this strategy, the player doubles the bet after every loss, such that the first win recovers all previous losses and adds a gain equal to the amount of the original bet. For instance, consider using this system to bet on black color in roulette with an initial bet of $1. Assume the first spin results in red or green color. Then, until the ball falls into a black pocket, you would bet: $2 on the second spin, $4 on the third spin, $8 on the fourth spin, etc. (note that the corresponding total amounts you would bet are $3 on the second spin, $7 on the third spin, $15 on the fourth spin, etc.). Suppose you decide to use the martingale betting system to bet on black color in an (unbiased) American roulette, starting with an $1 bet. Suppose also that you have a total of $63 available, and therefore you will run out of money after 6 consecutive spins resulting in either red or green color. What is your expected profit?

Solution:
The possible values for the profit are $-63 and $1. The latter value arises if there is at least one spin among the first 6 roulette spins that results in black color. The former value arises if all 6 spins result in red or green color, such that you bet: $1 on first spin (total = $1), $2 on second spin (total = $3), $4 on third spin (total = $7), $8 on fourth spin (total = $15), $16 on fifth spin (total = $31), and $32 on sixth spin (total = $63) at which point you would run out of money.

The probability that the profit value is $-63 is equal to the probability that there is no outcome with black color in 6 spins, that is, \((20/38)^6 = 0.021256\). More formally, \(\Pr(\text{profit} = -$63) = \Pr(A_1 \cap ... \cap A_6)\), where for \(i = 1, ..., 6\), \(A_i\) is the event that the \(i\)-th spin results in red or green color; therefore, \(\Pr(A_1 \cap ... \cap A_6) = \Pr(A_1) \times ... \times \Pr(A_6) = (20/38)^6\), assuming independent roulette spins. Moreover, \(\Pr(\text{profit} = $1) = \Pr(\text{at least one black in 6 spins}) = 1 - \Pr(\text{no black in 6 spins}) = 1 - 0.021256 = 0.978744\).

Therefore, \(E(\text{profit}) = (1 \times 0.978744) + ((-63) \times 0.021256) = -$0.3604\).