MATLAB Problems

1) a. By defining the matrix, say A (see matrix below) in MATLAB and using the command rref(A), we get the reduced row echelon form of the matrix A and the last column is the solution consisting of the coefficients $a_0, ... a_5$. Hence, $p(t) = 1.7125t - 1.1948t^2 + 0.6615t^3 - 0.0701t^4 + 0.0026t^5$ and so $p(7.5) = 64.6$ hundred lb.

b. If a polynomial of lower degree was used, then the resulting system of equations would be overdetermined - meaning it would have more equations than variables. Recall from lecture than an overdetermined system can have zero, one, or infinitely many solutions. In this case, zero solutions would arise if we tried to fit data that did not correspond to any cubic at the sampled points (e.g. if the function we are trying to fit is a quintic (5th order) polynomial, then we cannot find a cubic that will pass through six points in general). Similarly, infinitely many solutions will arise if we plugged in equations for the same point enough times to have less than four independent equations. For example, there are several different cubic equations that will pass through just two distinct points. Finally, we could have a unique solution if the data we were trying to fit was already a linear equation (or a quadratic, or a cubic), since the extra equations would not be independent. What we really want in curve fitting is to have a single, unique solution in general (which is not the case above, we only get unique solutions under special circumstances). This means that we need the same number of variables as unknowns (and that the points we are using are all distinct). Thus four independent equations is enough to ensure a unique cubic polynomial fit. Hence, in order to fully constrain a cubic polynomial we would need four points.
2) a. In order to create a square random matrix with \( n = 100 \) we write: 
\[
A = \text{rand}(100);
\]
then in order to see the computation time we can type 
\[
tic; \quad \text{rref}(A); \quad \text{toc}.
\]
Or we can write it us 
\[
tic; \quad \text{rref(rand(100)); \quad toc}.
\]
This gives approximately 0.28s. depending on the computer.

b. We do the same in order to find matrices bigger than \( n = 100 \) e.g. \( A = \text{rand}(120); \) for \( n = 120 \) matrix etc. Hence, we have 
\[
tic; \quad \text{rref(rand(120)); \quad toc},
\]
\[
tic; \quad \text{rref(rand(140)); \quad toc}, \quad \text{tic;
\quad } \text{rref(rand(160)); \quad toc}, \quad \text{tic;} \quad \text{rref(rand(180)); \quad toc}, \quad \text{tic; \quad } \text{rref(rand(200)); \quad toc}.
\]
We end up with

<table>
<thead>
<tr>
<th>n</th>
<th>runtime(sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.28</td>
</tr>
<tr>
<td>120</td>
<td>0.39</td>
</tr>
<tr>
<td>140</td>
<td>0.54</td>
</tr>
<tr>
<td>160</td>
<td>0.68</td>
</tr>
<tr>
<td>180</td>
<td>0.86</td>
</tr>
<tr>
<td>200</td>
<td>1.07</td>
</tr>
</tbody>
</table>

Note: We could use a for loop to do this computation instead of doing it separately for each case, we will probably see that later on.

c. We found that for \( n = 200 \) the computation time was approximately 1 second. One day consists of 86,400 sec., so since it is of order \( O(n^3) \), we get that 
\[
n = 200 \times \sqrt[3]{86,400} \rightarrow n \approx 9000
\]
in order for the \( \text{rref(rand(n))} \) to
Written Problems

Chapter 1.1

19. \[
\begin{bmatrix}
1 & h & 4 \\
3 & 6 & 8
\end{bmatrix}
\sim
\begin{bmatrix}
1 & h & 4 \\
0 & 6 - 3h & -4
\end{bmatrix}
\]
Write \( c \) for \( 6 - 3h \). If \( c = 0 \), that is, if \( h = 2 \), then the system has no solution, because 0 cannot equal –4. Otherwise, when \( h \neq 2 \), the system has a solution.

24. a. False. The definition of \textit{row equivalent} requires that there exist a sequence of row operations that transforms one matrix into the other.
   
b. True. See the box preceding the subsection titled \textit{Existence and Uniqueness Questions}.
   
c. False. The definition of \textit{equivalent systems} is in the second paragraph after equation (2).
   
d. True. By definition, a consistent system has \textit{at least one solution}.
Chapter 1.2

\[
\begin{bmatrix}
1 & 2 & 4 & 8 \\
2 & 4 & 6 & 8 \\
3 & 6 & 9 & 12
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 4 & 8 \\
0 & 0 & -2 & -8 \\
0 & 0 & -3 & -12
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 4 & 8 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Pivot cols 1 and 3.

\[
\begin{bmatrix}
1 & 2 & 4 & 8 \\
2 & 4 & 6 & 8 \\
3 & 6 & 9 & 12
\end{bmatrix}
\]
Chapter 1.3

11. The question

Is \( b \) a linear combination of \( a_1 \), \( a_2 \), and \( a_3 \)?

is equivalent to the question

Does the vector equation \( x_1a_1 + x_2a_2 + x_3a_3 = b \) have a solution?

The equation

\[
\begin{bmatrix}
1 & 0 & 5 & 2 \\
-2 & 1 & -6 & -1 \\
0 & 2 & 8 & 6
\end{bmatrix}
\]

has the same solution set as the linear system whose augmented matrix is

\[
M = \begin{bmatrix}
1 & 0 & 5 & 2 \\
-2 & 1 & -6 & -1 \\
0 & 2 & 8 & 6
\end{bmatrix}
\]

Row reduce \( M \) until the pivot positions are visible:

\[
M \sim \begin{bmatrix}
1 & 0 & 5 & 2 \\
0 & 1 & 4 & 3 \\
0 & 2 & 8 & 6
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 5 & 2 \\
0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The linear system corresponding to \( M \) has a solution, so the vector equation (*) has a solution, and therefore \( b \) is a linear combination of \( a_1 \), \( a_2 \), and \( a_3 \).

18. Some likely choices are \( 0 \cdot v_1 + 0 \cdot v_2 = 0 \), and

\[
1 \cdot v_1 + 0 \cdot v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad 0 \cdot v_1 + 1 \cdot v_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad 1 \cdot v_1 + 1 \cdot v_2 = \begin{bmatrix} -1 \\ -4 \end{bmatrix}, \quad 1 \cdot v_1 - 1 \cdot v_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}
\]
Chapter 1.4

17. Row reduction shows that only three rows of $A$ contain a pivot position:

$$
A = \begin{bmatrix}
1 & 3 & 0 & 3 \\
-1 & -1 & -1 & 1 \\
0 & -4 & 2 & -8 \\
2 & 0 & 3 & -1 \\
0 & -6 & 3 & -7 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 3 & 0 & 3 \\
0 & 2 & -1 & 4 \\
0 & -4 & 2 & -8 \\
0 & 0 & 0 & 0 \\
0 & 0 & 6 & -7 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 3 & 0 & 3 \\
0 & 2 & -1 & 4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Because not every row of $A$ contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $Ax = b$ does not have a solution for each $b$ in $\mathbb{R}^4$.

18. Row reduction shows that only three rows of $B$ contain a pivot position:

$$
B = \begin{bmatrix}
1 & 4 & 1 & 2 \\
0 & 1 & 3 & -4 \\
0 & 2 & -6 & 7 \\
2 & 9 & 5 & -7 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 4 & 1 & 2 \\
0 & 1 & 3 & -4 \\
0 & 2 & 6 & 7 \\
0 & 1 & 3 & -11 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 4 & 1 & 2 \\
0 & 1 & 3 & -4 \\
0 & 0 & 15 & 15 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Because not every row of $B$ contains a pivot position, Theorem 4 in Section 1.4 shows that not all vectors in $\mathbb{R}^4$ can be written as a linear combination of the columns of $B$. The columns of $B$ certainly do not span $\mathbb{R}^4$, because each column of $B$ is in $\mathbb{R}^4$, not $\mathbb{R}^3$.

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbb{R}^4$ can be written as a linear combination of the columns of $A$. Also, the columns of $A$ do not span $\mathbb{R}^4$.

20. The work in Exercise 18 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, the equation $Bx = y$ does not have a solution for each $y$ in $\mathbb{R}^4$, and the columns of $B$ do not span $\mathbb{R}^4$. 

6
26. The equation in \(x_1\) and \(x_2\) involves the vectors \(u\), \(v\), and \(w\), and it may be viewed as

\[
[u \ v] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = w.
\]

By definition of a matrix-vector product, \(x_1 u + x_2 v = w\). The stated fact that

\(2u - 3v - w = 0\)

can be rewritten as \(2u - 3v = w\). So, a solution is \(x_1 = 2\), \(x_2 = -3\).

Chapter 1.5

6. \[
\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

\(x_3 = 0\)

\(x_4 = x_3 = 0\). The variable \(x_3\) is free, \(x_1 = x_3\), and \(x_2 = x_3\).

\(0 = 0\)

In parametric vector form, the general solution is \(x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.

29. a. When \(A\) is a 4x4 matrix with three pivot positions, the equation \(Ax = 0\) has three basic variables and one free variable. So \(Ax = 0\) has a nontrivial solution.

b. With only three pivot positions, \(A\) cannot have a pivot in every row, so by Theorem 4 in Section 1.4, the equation \(Ax = b\) cannot have a solution for every possible \(b\) (in \(R^4\)).

30. a. When \(A\) is a 2x5 matrix with two pivot positions, the equation \(Ax = 0\) has two basic variables and three free variables. So \(Ax = 0\) has a nontrivial solution.

b. With two pivot positions and only two rows, \(A\) has a pivot position in every row. By Theorem 4 in Section 1.4, the equation \(Ax = b\) has a solution for every possible \(b\) (in \(R^5\)).
Chapter 1.7

16. The set is linearly dependent because the second vector is \(-3/2\) times the first vector.

23. \[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

24. \[
\begin{bmatrix}
0 & * \\
0 & *
\end{bmatrix}
\]

30. a. \(n\)

b. The columns of \(A\) are linearly independent if and only if the equation \(Ax = 0\) has only the trivial solution. This happens if and only if \(Ax = 0\) has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of \(A\) is a pivot column.