Answers - Homework 2

Written Problems

Chapter 1.8

13. A reflection through the origin.


The transformation in Exercise 13 may also be described as a rotation of $\pi$ radians about the origin or a rotation of $-\pi$ radians about the origin.

15. A reflection through the line $x_2 = x_1$.

16. A scaling by a factor of 2 and a projection onto the $x_2$ axis.

17. $T(2u) = 2T(u) = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$, $T(3v) = 3T(v) = 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$, and

$T(2u + 3v) = 2T(u) + 3T(v) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$. 
25. Any point \( x \) on the line through \( p \) in the direction of \( v \) satisfies the parametric equation
\[
x = p + tv
\]
for some value of \( t \). By linearity, the image \( T(x) \) satisfies the parametric equation
\[
T(x) = T(p + tv) = T(p) + tT(v)
\]
(*)
If \( T(v) = 0 \), then \( T(x) = T(p) \) for all values of \( t \), and the image of the original line is just a single point. Otherwise, (*) is the parametric equation of a line through \( T(p) \) in the direction of \( T(v) \).

31. Suppose that \( \{v_1, v_2, v_3\} \) is linearly dependent. Then there exist scalars \( c_1, c_2, c_3 \), not all zero, such that
\[
c_1v_1 + c_2v_2 + c_3v_3 = 0
\]
Then \( T(c_1v_1 + c_2v_2 + c_3v_3) = T(0) = 0 \). Since \( T \) is linear,
\[
c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0
\]
Since not all the weights are zero, \( \{T(v_1), T(v_2), T(v_3)\} \) is a linearly dependent set.

Chapter 1.9

17. To express \( T(x) \) as \( Ax \), write \( T(x) \) and \( x \) as column vectors, and then fill in the entries in \( A \) by inspection, as done in Exercises 15 and 16. Note that since \( T(x) \) and \( x \) have four entries, \( A \) must be a 4×4 matrix.
\[
T(x) = \begin{bmatrix}
x_1 + 2x_2 \\
0 \\
2x_2 + x_3 \\
x_2 - x_4
\end{bmatrix} =
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 \\
0 & 1 & 0 & -1
\end{bmatrix}
\]

26. The standard matrix \( A \) of the transformation \( T \) in Exercise 2 is 2×3. Its columns are linearly dependent because \( A \) has more columns than rows. So \( T \) is not one-to-one, by Theorem 12. Also, \( A \) is row equivalent to \( \begin{bmatrix}1 & -2 & 3 \\ 0 & 17 & -20\end{bmatrix} \), which shows that the rows of \( A \) span \( R^2 \). By Theorem 12, \( T \) maps \( R^2 \) onto \( R^2 \).

31. “\( T \) is one-to-one if and only if \( A \) has \( n \) pivot columns.” By Theorem 12(b), \( T \) is one-to-one if and only if the columns of \( A \) are linearly independent. And from the statement in Exercise 30 in Section 1.7, the columns of \( A \) are linearly independent if and only if \( A \) has \( n \) pivot columns.
Chapter 2.1

35. If \( T: \mathbb{R}^n \to \mathbb{R}^m \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^m \), then its standard matrix \( A \) has a pivot in each row, by Theorem 12 and by Theorem 4 in Section 1.4. So \( A \) must have at least as many columns as rows. That is, \( m \leq n \). When \( T \) is one-to-one, \( A \) must have a pivot in each column, by Theorem 12, so \( m \geq n \).

10. \( AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}, AC = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix} \)

13. Use the definition of \( AB \) written in reverse order: \( [Ab_1 \cdots Ab_n] = A[b_1 \cdots b_n] \). Thus \( [Qr_1 \cdots Qr_j] = QR \), when \( R = [r_1 \cdots r_j] \).

18. The third column of \( AB \) is also all zeros because \( Ab_3 = A0 = 0 \)

20. The first two columns of \( AB \) are \( Ab_1 \) and \( Ab_2 \). They are equal since \( b_1 \) and \( b_2 \) are equal.

27. The product \( u^T v \) is a 1\times1 matrix, which usually is identified with a real number and is written without the matrix brackets.

\[
\begin{align*}
u^T v &= \begin{bmatrix} -3 & 2 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -3a + 2b - 5c, \\
v^T u &= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix} = -3a + 2b - 5c
\end{align*}
\]

\[
\begin{align*}
u v^T &= \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{bmatrix}, \\
vv^T &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{bmatrix}
\end{align*}
\]
Chapter 2.4

3. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the left.

\[
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
0A + IC & 0B + ID \\
IA + 0C & IB + 0D
\end{bmatrix}
= \begin{bmatrix}
C & D \\
A & B
\end{bmatrix}
\]

12. a. False. The both $AB$ and $BA$ are defined.
b. False. The $R^T$ and $Q^T$ also need to be switched.

15. The column-row expansions of $G_k$ and $G_{k+1}$ are:

$G_k = X_k X_k^T$

$= \text{col}_i(X_k) \text{row}_i(X_k^T) + \ldots + \text{col}_j(X_k) \text{row}_j(X_k^T)$

and

$G_{k+1} = X_{k+1} X_{k+1}^T$

$= \text{col}_i(X_{k+1}) \text{row}_i(X_{k+1}^T) + \ldots + \text{col}_j(X_{k+1}) \text{row}_j(X_{k+1}^T) + \text{col}_i(X_k) \text{row}_i(X_k^T) + \text{col}_i(X_{k+1}) \text{row}_i(X_{k+1}^T)$

$= G_k + \text{col}_i(X_{k+1}) \text{row}_i(X_k^T)$

since the first $k$ columns of $X_{k+1}$ are identical to the first $k$ columns of $X_k$. Thus to update $G_k$ to produce $G_{k+1}$, the matrix $\text{col}_i(X_{k+1}) \text{row}_i(X_k^T)$ should be added to $G_k$. 
Matlab Problems

Chapter 1.6, Problem 14

We can use, as we have already learned from the MATLAB section, the \textit{rref()} command in order to row reduce the matrix below.

Write the equations for each intersection:

<table>
<thead>
<tr>
<th>Intersection</th>
<th>Flow in</th>
<th>Flow out</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>80</td>
<td>(x_1 + x_5)</td>
</tr>
<tr>
<td>B</td>
<td>(x_1 + x_2 + 100)</td>
<td>(x_4)</td>
</tr>
<tr>
<td>C</td>
<td>(x_3)</td>
<td>(x_2 + 90)</td>
</tr>
<tr>
<td>D</td>
<td>(x_4 + x_5)</td>
<td>(x_3 + 90)</td>
</tr>
</tbody>
</table>

Rearrange the equations:

\[
\begin{align*}
  x_1 + x_5 &= 80 \\
  x_1 + x_2 - x_4 &= -100 \\
  x_2 - x_3 &= -90 \\
  x_3 - x_4 - x_5 &= -90
\end{align*}
\]

Reduce the augmented matrix:

\[
\begin{bmatrix}
  1 & 0 & 0 & 0 & 1 & 80 \\
  1 & 1 & 0 & -1 & 0 & -100 \\
  0 & 1 & -1 & 0 & 0 & -90 \\
  0 & 0 & 1 & -1 & -1 & -90
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 0 & 0 & 0 & 1 & 80 \\
  0 & 1 & 0 & -1 & -1 & -180 \\
  0 & 0 & 1 & -1 & -1 & -90 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

a. The general solution is

\[
\begin{aligned}
  x_1 &= 80 - x_5 \\
  x_2 &= x_4 + x_5 - 180 \\
  x_3 &= x_4 + x_5 - 90 \\
  x_4 &\text{ is free} \\
  x_5 &\text{ is free}
\end{aligned}
\]

b. If \(x_5 = 0\), then the general solution is

\[
\begin{aligned}
  x_1 &= 80 \\
  x_2 &= x_4 - 180 \\
  x_3 &= x_4 - 90 \\
  x_4 &\text{ is free}
\end{aligned}
\]

c. Since \(x_2\) cannot be negative, the minimum value of \(x_4\) when \(x_5 = 0\) is 180.
We can write $x_1 = M \times x_0$ in MATLAB and multiply matrix $M$ with the column vector $x_0$ in order to find $x_1$. We can use the recurrence relation $x_{k+1} = Mx_k$ to find $x_2, ..., x_{20}$ by typing $x_2 = M \times x_1$. But $x_2 = M \times (M \times x_0) = M^2 \times x_0, ..., x_{20} = M^{20} \times x_0$ etc. in MATLAB and we get the results below.

13. [M] The order of entries in a column of a migration matrix must match the order of the columns. For instance, if the first column concerns the population in the city, then the first entry in each column must be the fraction of the population that moves to (or remains in) the city. In this case, the data in the exercise leads to $M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix}$.

a. Some of the population vectors are

$x_1 = \begin{bmatrix} 523,293 \\ 476,707 \end{bmatrix}$, $x_{10} = \begin{bmatrix} 472,737 \\ 527,263 \end{bmatrix}$, $x_{15} = \begin{bmatrix} 439,417 \\ 560,583 \end{bmatrix}$, $x_{20} = \begin{bmatrix} 417,456 \\ 582,544 \end{bmatrix}$

The data here shows that the city population is declining and the suburban population is increasing, but the changes in population each year seem to grow smaller.

b. When $x_0 = \begin{bmatrix} 350,000 \\ 650,000 \end{bmatrix}$, the situation is different. Now

$x_1 = \begin{bmatrix} 358,523 \\ 641,477 \end{bmatrix}$, $x_{10} = \begin{bmatrix} 364,140 \\ 635,860 \end{bmatrix}$, $x_{15} = \begin{bmatrix} 367,843 \\ 632,157 \end{bmatrix}$, $x_{20} = \begin{bmatrix} 370,283 \\ 629,717 \end{bmatrix}$

The city population is increasing slowly and the suburban population is decreasing. No other conclusions are expected. (This example will be analyzed in greater detail later in the text.)
Chapter 2.1, Problem 37

We have already learned that we can create random matrices in MATLAB by typing the command `rand()`. In this case, we will type $A = rand(4)$ and $B = rand(4)$ for square matrices $A, B$ $4 \times 4$ and then we will type $A \ast B$ and $B \ast A$. We can notice that $AB \neq BA$ in most cases, since the equality $AB = BA$ is very likely to be false in $4 \times 4$ random matrices. Generally $AB \neq BA$. However, notice that the element multiplication gives $A. \ast B = B. \ast A$. 