AMS 10 - Fall 2014

Problem Set 6
Due: Friday 11/14/14 - 5pm (Homework Box #6 in Jack's Lounge)

1 Written problems

§4.1 - 5, 6, 24
§4.2 - 4, 17, 19, 23
§4.3 - 1, 2, 21, 23
§4.4 - 1, 2, 13, 16
§4.5 - 3, 4, 13, 21
§4.6 - 2, 9, 15

2 Matlab problems

2.1 Visualizing the column space

In this question you'll determine visually whether vectors lie in the column space of a matrix or not. For the entire question, you'll be using a single $3 \times 2$ matrix, $A$, which you can generate with the command $A = \text{rand}(3,2)$.

(a) First, you'll need variables for the two columns of this matrix. Define $u = A(:,1)$, $v = A(:,2)$ as the column vectors of $A$. You can graph the column space of $A$ using the commands

$$[s,t] = \text{meshgrid}((-1:0.1:1), (-1:0.1:1));$$
$$X = s*u(1) + t*v(1); Y = s*u(2) + t*v(2); Z = s*u(3) + t*v(3);$$
$$\text{surf}(X,Y,Z); \text{axis square}; \text{colormap hot}; \text{hold on}$$

A graph should appear in a separate window showing $\text{Col}(A)$. From the Tools menu choose the command Rotate 3D. Then you can click and drag on the plot to rotate the graph in three dimensions. You don't need to print out anything for this section, this plot will be included with your plot for the next part.

(b) Generate a random vector in $\mathbb{R}^3$ with

$b = \text{rand}(3,1)$

You can graph $\text{span}(b)$ in the same figure as $\text{Col}(A)$ using

$$r = [-1:0.05:1];$$
$$\text{plot3}(r*b(1), r*b(2), r*b(3), '+')$$

Determine whether $b$ lies in $\text{Col}(A)$ graphically. By rotating it enough, you should be able to see whether the entire line $\text{span}(b)$ lies in $\text{Col}(A)$ or not. Note: For every vector $v$, the line $\text{span}(v)$ will intersect $\text{Col}(A)$ in the point $0$, since every subspace contains the zero vector. You must look to see if all of the line through your vector $b$ is in $\text{Col}(A)$. (Hint: Try to make $\text{Col}(A)$ look like a line by viewing it edge-on.)

Print the graph with a good choice of rotation showing whether or not $b$ is in $\text{Col}(A)$.
(c) From your analysis in the previous part, is there a vector $x \in \mathbb{R}^2$ such that $Ax = b$ for the matrix $A$ and vector $b$ you used?

(d) Generate a random vector $c$ lying in Col($A$) by taking a random linear combination of the columns of $A$ using the commands

$$z = \text{rand}(2,1), \quad c = A*z$$

Plot a new graph of Span($c$) and Col($A$) using

```matlab
figure, surf(X,Y,Z); axis square; colormap hot, hold on
plot3(r*c(1),r*c(2),r*c(3), '+')
```

Rotate your figure as in (b) to show that the entire line Span($c$) is contained in Col($A$). After making a good choice of rotation print the graph and include it with 

2.2 Reduced row echelon form and the null space

For this problem, we'll need a partially random matrix $A$ which we'll construct from linear combinations of the columns of a random $3 \times 3$ matrix, $B$. First generate a random $3 \times 3$ matrix, and check to make sure it has full pivots (i.e. its rank is 3) with

$$B = \text{rand}(3,3), \quad \text{rank}(B)$$

Since $B$ is random, it is very likely to have rank 3. If it doesn’t, then re-generate it until its rank is 3.

Now we know that that columns of $B$ are linearly independent, since there’s a pivot in every column. Additionally, we know that this means that Col($B$) = $\mathbb{R}^3$, any vector in $\mathbb{R}^3$ can be written as linear combinations of the columns of $B$.

We’ll now define a matrix $A$ whose columns are linear combinations of the columns of $B$, using

$$A = [B(:,1), B(:,2), 2*B(:,1) + 3*B(:,2), 4*B(:,1) - 5*B(:,2), B(:,3)]$$

We now have 5 columns that live in $\mathbb{R}^3$, so they must be linearly dependent if taken all together. To check this, compute the row-reduced echelon format of $A$,

$$R = \text{rref}(A)$$

(a) Which columns are pivot columns of $A$ and $R$?

(b) What are the free variables? From this, what is the dimension of the null space of $A$?

(c) Why is column #3 of $R$ always

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \text{and column #4 of } R \text{ always } \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}$$

(d) Use the function provided to you in nulbasis.m to calculate the non-trivial solutions to the equation $Ax = 0$ via

$$N = \text{nulbasis}(A)$$

The columns of $N$ are each solutions to $Ax = 0$, each obtained by setting one free variable equal to one and the others to zero - these are just the vectors you get when writing the solution to $Ax = 0$ in parametric form (see example 3 in section 4.2, for another example of this).

(e) Finally, generate a random linear combination of the columns of $N$,

$$s = \text{rand}(1), \quad t = \text{rand}(1), \quad x=s*N(:,1) + t*N(:,2)$$

and check that they also satisfy $Ax = 0$ and $Rx = 0$. They should because the Nul($A$) is a vector space, so any linear combination of vectors in the null space is also in the null space.