Answers - Homework 6

1. Written Problems

Chapter 4.1

5. Yes. Since the set is Span \( \{e^2\} \), the set is a subspace by Theorem 1.

6. No. The zero vector is not in the set.

24. a. True. See the definition of a vector space.
b. True. See statement (3) in the box before Example 1.
c. True. See the paragraph before Example 6.
d. False. See Example 8.
e. False. The second and third parts of the conditions are stated incorrectly. For example, part (ii) does not state that \( u \) and \( v \) represent all possible elements of \( H \).

Chapter 4.2

4. First find the general solution of \( Ax = 0 \) in terms of the free variables. Since

\[
\begin{bmatrix}
A & 0 \\
\end{bmatrix} =
\begin{bmatrix}
1 & -3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix},
\]

the general solution is \( x_i = 3x_2 \), \( x_3 = 0 \), with \( x_2 \) and \( x_4 \) free. So

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
\]

and a spanning set for Nul \( A \) is

\[
\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\
0 \\ 0 \\ 0 \\ 1 \\
\end{bmatrix} \right\}.
\]
17. The matrix $A$ is a $4 \times 2$ matrix. Thus
   (a) $\text{Nul } A$ is a subspace of $\mathbb{R}^4$, and
   (b) $\text{Col } A$ is a subspace of $\mathbb{R}^2$.

18. The matrix $A$ is a $4 \times 3$ matrix. Thus
   (a) $\text{Nul } A$ is a subspace of $\mathbb{R}^1$, and
   (b) $\text{Col } A$ is a subspace of $\mathbb{R}^4$.

19. The matrix $A$ is a $2 \times 5$ matrix. Thus
   (a) $\text{Nul } A$ is a subspace of $\mathbb{R}^5$, and
   (b) $\text{Col } A$ is a subspace of $\mathbb{R}^2$.

23. Consider the system with augmented matrix $[ A \quad w ]$. Since

\[
\begin{bmatrix}
A & w
\end{bmatrix} = \begin{bmatrix}
1 & -2 & -1 \\
0 & 0 & 0
\end{bmatrix},
\]

the system is consistent and $w$ is in $\text{Col } A$. Also, since

\[
Aw = \begin{bmatrix}
-2 \\
-1
\end{bmatrix} \begin{bmatrix}
2 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

$w$ is in $\text{Nul } A$.

Chapter 4.3

1. Consider the matrix whose columns are the given set of vectors. This $3 \times 3$ matrix is in echelon form, and has 3 pivot positions. Thus by the Invertible Matrix Theorem, its columns are linearly independent and span $\mathbb{R}^3$. So the given set of vectors is a basis for $\mathbb{R}^3$.

2. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. This $3 \times 3$ matrix has only 2 pivot positions. Thus by the Invertible Matrix Theorem, its columns do not span $\mathbb{R}^3$.

21. a. False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.
   b. False. The set $\{b_1, \ldots, b_p\}$ must also be linearly independent. See the definition of a basis.
   c. True. See Example 3.
23. Let \( A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \). Then \( A \) is square and its columns span \( \mathbb{R}^4 \) since \( \mathbb{R}^3 = \text{Span}(v_1, v_2, v_3, v_4) \). So its columns are linearly independent by the Invertible Matrix Theorem, and \( \{v_1, v_2, v_3, v_4\} \) is a basis for \( \mathbb{R}^4 \).

**Chapter 4.4**

1. We calculate that
   \[
   x = 5 \begin{bmatrix} 3 \\ -5 \\ 3 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.
   \]

2. We calculate that
   \[
   x = (-2) \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -26 \\ 1 \end{bmatrix}.
   \]

13. We must find \( c_1, c_2, \) and \( c_3 \) such that
   \[
   c_1 (1 + t^2) + c_2 (t + t^3) + c_3 (1 + 2t + t^2) = p(t) = 1 + 4t + 7t^2.
   \]
   Equating the coefficients of the two polynomials produces the system of equations
   \[
   \begin{align*}
   c_1 + c_1 &= 1 \\
   c_2 + 2c_3 &= 4 \\
   c_1 + c_2 + c_3 &= 7
   \end{align*}
   \]
   We row reduce the augmented matrix for the system of equations to find
   \[
   \begin{bmatrix}
   1 & 0 & 1 & 1 \\
   0 & 1 & 2 & 4 \\
   1 & 1 & 1 & 7
   \end{bmatrix}
   \]
   After row reduction we have
   \[
   [p]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.
   \]
   One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis \( \{1, t, t^2\} \); the same system of linear equations results.

   
   b. False. By definition, the coordinate mapping goes in the opposite direction.
   
   c. True. If the plane passes through the origin, as in Example 7, the plane is isomorphic to \( \mathbb{R}^2 \).
3. This subspace is \( \mathcal{H} = \text{Span}\{v_1, v_2, v_3\} \), where \( v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \), \( v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} \), and \( v_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \). Theorem 4 in Section 4.3 can be used to show that this set is linearly independent: \( v_1 \neq 0 \), \( v_2 \) is not a multiple of \( v_1 \), and (since its first entry is not zero) \( v_3 \) is not a linear combination of \( v_1 \) and \( v_2 \). Thus \( \{v_1, v_2, v_3\} \) is linearly independent and is thus a basis for \( H \). Alternatively, one can show that this set is linearly independent by row reducing the matrix \( \begin{bmatrix} v_1 & v_2 & v_3 & 0 \end{bmatrix} \). Hence the dimension of the subspace is 3.

4. This subspace is \( \mathcal{H} = \text{Span}\{v_1, v_2\} \), where \( v_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \). Since \( v_1 \) and \( v_2 \) are not multiples of each other, \( \{v_1, v_2\} \) is linearly independent and is thus a basis for \( H \). Hence the dimension of \( H \) is 2.

13. The matrix \( A \) is in echelon form. There are three pivot columns, so the dimension of \( \text{Col} \ A \) is 3. There are two columns without pivots, so the equation \( Ax = 0 \) has two free variables. Thus the dimension of \( \text{Nul} \ A \) is 2.
21. The matrix whose columns are the coordinate vectors of the Hermite polynomials relative to the standard basis \( \{1, t, t^2, t^3\} \) of \( \mathbb{P}_3 \) is
   \[
   A = \begin{bmatrix}
   1 & 0 & -2 & 0 \\
   0 & 2 & 0 & -12 \\
   0 & 0 & 4 & 0 \\
   0 & 0 & 0 & 8 \\
   \end{bmatrix}.
   \]
   This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Hermite polynomials themselves are linearly independent in \( \mathbb{P}_3 \). Since there are four Hermite polynomials and \( \dim \mathbb{P}_3 = 4 \), the Basis Theorem states that the Hermite polynomials form a basis for \( \mathbb{P}_3 \).

Chapter 4.6

2. The matrix \( B \) is in echelon form. There are three pivot columns, so the dimension of \( \text{Col} A \) is 3. There are three pivot rows, so the dimension of \( \text{Row} A \) is 3. There are two columns without pivots, so the equation \( A\mathbf{x} = \mathbf{0} \) has two free variables. Thus the dimension of \( \text{Nul} A \) is 2. A basis for \( \text{Col} A \) is the pivot columns
   \[
   \begin{pmatrix}
   1 \\
   2 \\
   3 \\
   \end{pmatrix},
   \begin{pmatrix}
   4 \\
   6 \\
   3 \\
   \end{pmatrix},
   \begin{pmatrix}
   2 \\
   -3 \\
   -3 \\
   \end{pmatrix}.
   \]
   A basis for \( \text{Row} A \) is the pivot rows of \( B \): \( \{(1,3,4,-1,2),(0,0,1,-1,1),(0,0,0,0,-5)\} \). To find a basis for \( \text{Nul} A \) row reduce to reduced echelon form:
   \[
   A = \begin{bmatrix}
   1 & 3 & 0 & 3 & 0 \\
   0 & 0 & 1 & -1 & 0 \\
   0 & 0 & 0 & 0 & 1 \\
   \end{bmatrix},
   \]
   The solution to \( A\mathbf{x} = \mathbf{0} \) in terms of free variables is \( x_1 = -3x_2 - 3x_4, x_3 = x_4, x_4 = 0 \), with \( x_2 \) and \( x_4 \) free. Thus a basis for \( \text{Nul} A \) is
   \[
   \begin{pmatrix}
   -3 \\
   1 \\
   0 \\
   \end{pmatrix},
   \begin{pmatrix}
   -3 \\
   0 \\
   1 \\
   \end{pmatrix},
   \begin{pmatrix}
   0 \\
   0 \\
   1 \\
   \end{pmatrix}.
   \]

9. Since \( \dim \text{Nul} A = 3 \), \( \text{rank} A = 6 - \dim \text{Nul} A = 6 - 3 = 3 \). So \( \dim \text{Col} A = \text{rank} A = 3 \).
   No, \( \text{Col} A \subseteq \mathbb{R}^3 \). It is true that \( \dim \text{Col} A = \text{rank} A = 3 \), but \( \text{Col} A \) is a subspace of \( \mathbb{R}^4 \).
15. Since the rank of $A$ equals the number of pivot positions which the matrix has, and $A$ could have at most 3 pivot positions, rank $A \leq 3$. Thus dimNul $A = 7 - \text{rank } A = 7 - 3 = 4$.

2. MATLAB Problems
Problem 1

a) By typing

clear all, close all;

A = rand(3,2);
u = A(:,1);
v = A(:,2);

[s,t] = meshgrid((-1:0.1:1),(-1:0.1:1));

X = s*u(1) + t*v(1);
Y = s*u(2) + t*v(2);
Z = s*u(3) + t*v(3);

surf(X,Y,Z); axis square; colormap hot; hold on

in the Command Window or saving it as a script (m-file) in MATLAB, we can create the 3D plot of \text{Col}(A)(see Figure 1).

b) We can add the MATLAB commands below to our previous script. By doing so, we create a random vector \(b\) and we can graphically check if this vector lies in \text{Col}(A).

\[b = \text{rand}(3,1);\]
\[r = -1:0.05:1;\]
\[\text{plot3}(r*b(1), r*b(2), r*b(3),'.')\]

It is obvious from the 3D plot (see Figure 2), that \text{span}(b) does not lie in \text{Col}(A) but it intersects \text{Col}(A) at zero, since every subspace contains the
zero vector. Basically, it is very improbable, if not impossible, for a random vector \( b \) to lie in \( \text{Col}(A) \) of a random matrix \( A \).

c) From part (b), we can see that \( b \) does not lie in \( \text{Col}(A) \) but it is intersects it. This is another way of asking the same thing as in part (b). Hence, from Figure 2 again, we can see that there cannot be any vector \( x \in \mathbb{R}^2 \) such that \( Ax = b \).

d) We again have to add the MATLAB commands below to our script in order to get another 3D plot. This time we can see that \( \text{Span}(c) \) does lie in \( \text{Col}(A) \) and it can easily be explained by the fact that it is a linear combination of the columns of matrix \( A \), as we created it that way. (see Figure 3)

\[
\begin{align*}
z &= \text{rand}(2,1); \\
c &= A*z; \\
\text{figure, surf(X,Y,Z); axis square, colormap cool, hold on} \\
\text{plot3(r*c(1),r*c(2),r*c(3),'r.'})
\end{align*}
\]
Figure 1: 3D plot of $\text{Col}(A)$
Figure 2: 3D plot of span(b) and Col(A)
Figure 3: 3D plot of span(c) and Col(A)
**Problem 2**

By writing the script below in MATLAB we can answer questions (a)-(e).

```matlab
clear all, close all;

% part (a) - (c)
B = rand(3,3); rankB = rank(B)
A = [B(:,1), B(:,2), 2*B(:,1) + 3*B(:,2), 4*B(:,1) - 5*B(:,2), B(:,3)];
R = rref(A)

% part (d)
N = nulbasis(A)

% part (e)
s = rand(1);
t = rand(1);
x = s*N(:,1) + t*N(:,2)

check = A*x  % we should get Ax=0
check1= R*x  % we should get Rx=0

a) We can see from the output matrix R which is R = rref(A), that the pivot columns are the first, the second and the last(fifth) column. (see R output)

b) The free variables are $x_3$ and $x_4$, so this means that the dimension of the null space of A is equal to 2, i.e. dim(null(A))=2. (see R output)

c) This is true because these two columns are linear combinations of the other columns of matrix A(and of matrix B).
d) By saving the nulbasis.m script and typing \texttt{N=nulbasis(A)} (see script), we can find the nontrivial solutions to the equation $Ax = 0$, meaning that we can find the null space of $A$, since basically the columns of $N$ are the nontrivial solutions of the homogeneous system $Ax = 0$. (see N output)

e) Finally, we can check that a random linear combination of the columns of $N$ satisfies again the system $Ax = 0$ as expected, since Nul(A) is a vector space, so any linear combination of vectors in the null space is also in the null space. (see output check to see that indeed $A^*x=0$ and check1 for $R^*x=0$). The output check in MATLAB does not give exactly zero for all the entries, however it gives a small number of order $10^{-15}$ and as we have explained in some other HW such a small number means technically zero, so we get the expected answer.
rankB =

    3

R =

    1.0000   0   2.0000   4.0000   0
    0   1.0000   3.0000  -5.0000   0
    0   0   0   0   0   1.0000

N =

    -2.0000  -4.0000
    -3.0000   5.0000
    1.0000   0
    0   1.0000
    0   0

x =

    -3.4527
    3.0849
    0.2238
    0.7513
    0

check =

    1.0e-015 *

    0.1388
    0
    0.0694

check1 =

    0
    0
    0