Announcements

- HW 2 graded (better!)
  - HW1 mean: 65%
  - HW2 mean: 75%

- Midterm in a week!
  - Multiple choice (like iClicker) + short answer (like HW)
  - Should be straight-forward if you can do HW
  - I'll give review problems on Friday
  - Review session TBA, looking like Tues 10/28 4:30-5:30
  - Extra office hours for me that Tues (1-3pm)
Suppose you have a two-sector economy: water and electricity production.

Producing 1 unit of electricity takes 0.2 units of electricity and 0.4 units of water.
Producing 1 unit of water takes 0.3 units of electricity and 0.1 units of water.

What is the consumption matrix for this economy?

A: \[
\begin{bmatrix}
0.2 & 0.4 \\
0.3 & 0.1
\end{bmatrix}
\]

B: \[
\begin{bmatrix}
0.2 & 0.3 \\
0.4 & 0.1
\end{bmatrix}
\]

C: \[
\begin{bmatrix}
0.2 & 0.4 \\
0.1 & 0.3
\end{bmatrix}
\]

Answer: B - The columns of the consumption matrix are the ones that are related to an economic sector. The stresses of producing one unit of electricity are 0.2 on electricity and 0.4 on water, which means that these values must appear in the same column. Note that the order is up to you when starting the problem, as long as you’re consistent throughout the problem.
Computer Graphics

Raster vs. Vector graphics

Raster

http://en.wikipedia.org/wiki/Raster_graphics

Vector

http://en.wikipedia.org/wiki/Vector_graphics
Computer Graphics

When would we use each?

- Raster graphics good for photos, movies, etc.
- Vector graphics good for logos, necessary for 3D games

For vector graphics, we need to store:

- What kind of shape it is
- Enough points & distances to specify the shape
- Style and color of the line
- Style and color of the interior
- etc.
Vector Graphics
Vector Graphics

Let’s represent the ship by a triangle, starting with its vertices at (0, 0), (2, 0), (1, 3).

We could represent this as a matrix of vertices, with the understanding (stored as an additional piece of data) that they are connected in order and the last connects to the first. 

$$\begin{bmatrix}
0 & 2 & 1 \\
0 & 0 & 3
\end{bmatrix}$$
Say we want to translate the ship by \( x_0 \) in the x direction and \( y_0 \) in the y direction. Would this be a linear transformation?

\[
\begin{bmatrix}
0 & 2 & 1 \\
0 & 0 & 3
\end{bmatrix} + 
\begin{bmatrix}
x_0 & x_0 & x_0 \\
y_0 & y_0 & y_0
\end{bmatrix}
\]

A: Yes

B: No, it violates \( T(a + b) = T(a) + T(b) \)

C: No, it violates \( T(ca) = cT(a) \)

D: No, it violates both

Answer: D - Let's pick an \( x_0 \) and \( y_0 \) to translate by - that determines our linear transformation, \( T \). \( T \) acts on vectors in \( \mathbb{R}^2 \) (or matrices who's columns are the vectors we want to transform) labeled \( a \) and \( b \), and produces vectors in \( \mathbb{R}^2 \) (or matrices who's columns are in \( \mathbb{R}^2 \)).

If we translate the sum of two vectors by \((x_0, y_0)\) then \( T(a + b) \) applies one translation, while \( T(a) + T(b) \) applies two translations - one for each vector. Thus \( T(a) + T(b) \) shifts the sum of the vectors by \((2x_0, 2y_0)\) rather than just \((x_0, y_0)\). Think about the case when \( a \) and \( b \) are the same initial vector here.

If our transformation translates all vectors by \((x_0, y_0)\), then multiplying the initial vector by a constant, \( c \), will not necessarily multiply the result by \( c \). The clearest counterexample is when \( c=0 \). In that case, \( ca = 0 \) and the translation shift the zero vector to \((x_0, y_0)\), which is of course not a multiple of \((0,0)\).
Translations

There’s a simple fix! Add another dimension of ones to each point - making it a vector in $\mathbb{R}^3$ instead of $\mathbb{R}^2$.

(this is called using *homogenous coordinates*)

\[
\begin{bmatrix}
0 & 2 & 1 \\
0 & 0 & 3
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 2 & 1 \\
0 & 0 & 3 \\
1 & 1 & 1
\end{bmatrix}
\]

Now we can write translations by $(x_0, y_0)$ as:

\[
\begin{bmatrix}
1 & 0 & x_0 \\
0 & 1 & y_0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 1 \\
0 & 0 & 3 \\
1 & 1 & 1
\end{bmatrix}
\]
Translations

Suppose we want to fly our ship two units in the direction it is currently facing (up). We need to translate by (0, 2):

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 1 \\
0 & 0 & 3 \\
1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 2 & 1 \\
2 & 2 & 5 \\
1 & 1 & 1
\end{bmatrix}
\]

The new vertices (0, 2), (2, 2), (1, 5) give us the new ship image:
Rotations

Turning our ship

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} \mapsto \begin{bmatrix}
x_1 \\
y_1
\end{bmatrix} = \begin{bmatrix}
\cos \phi \\
\sin \phi
\end{bmatrix}
\quad \begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix} \mapsto \begin{bmatrix}
x_2 \\
y_2
\end{bmatrix} = \begin{bmatrix}
-\sin \phi \\
\cos \phi
\end{bmatrix}
\]

Turning in Place

We'll crash into an asteroid if we can't change direction. Let's find the images of the columns of \( I_2 \) under counterclockwise rotation at an angle of \( \phi \).
Rotations

90 degree turn to the left:

\[
\begin{bmatrix}
  0 & -1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  0 & 2 & 1 \\
  0 & 0 & 3 \\
  1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 & -3 \\
  0 & 2 & 1 \\
  1 & 1 & 1
\end{bmatrix}
\]

Not quite what we wanted…
We want the ship to turn in place
Rotations

We want our rotation to be about the center of the ship, say, the point halfway between the front and back ends on the line straight back from the tip. In our beginning position that is (1, 1.5).

How should we do this?
Composite Transformations

Easiest way is to translate, rotate, and translate back

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1.5 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1.5 \\
0 & 0 & 1
\end{bmatrix}
\]

Translate by (1, 1.5)  Rotate left 90 degrees  Translate by (-1, -1.5)

\[
= \begin{bmatrix}
0 & -1 & 2.5 \\
1 & 0 & 0.5 \\
0 & 0 & 1
\end{bmatrix}
\]

Matrix that does all of this at once
Composite Transformations

Now our full transformation for rotating in place gives us:

\[
\begin{bmatrix}
0 & -1 & 2.5 \\
1 & 0 & 0.5 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 1 \\
0 & 0 & 3 \\
1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
2.5 & 2.5 & -0.5 \\
0.5 & 2.5 & 1.5 \\
1 & 1 & 1
\end{bmatrix}
\]

We get final vertices at (2.5, 0.5), (2.5, 2.5), and (−0.5, 1.5).
Let’s graph it:

booyah!