**Definition:** An $m \times n$ **matrix** is a rectangular array with $m$ rows and $n$ columns:

$$
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix} \leftarrow \text{row } i
$$

$\uparrow$

\text{column } j

The entry in row $i$ and column $j$ is denoted by $a_{ij}$. These entries can be real numbers or complex numbers, (or functions, etc.).
Comments:

1. The *size* of a matrix is given by its dimensions $m \times n$.

2. Two matrices $A$ and $B$ are equal if and only if they have the same size and $a_{ij} = b_{ij}$ for every $i$ and every $j$.

3. We denote by $\mathbb{R}^{m \times n}$ the set of all $m \times n$ matrices with real coefficients.

4. An $m \times 1$ matrix is called a *column vector*, and a $1 \times n$ matrix is called a *row vector*.

5. The space of all column vectors with $m$ rows and real coefficients is denoted by $\mathbb{R}^m$. Likewise we denote by $\mathbb{R}^n$ the space of all row vectors with $n$ columns (and real coefficients).

In other words, we use $\mathbb{R}^k$ to denote the space of vectors with $k$ entries and real coefficients. Whether these are represented as rows or columns, depends on the context.
**Definition:** A *submatrix* of a matrix $A$ is a matrix obtained from $A$ by deleting some columns and/or rows from $A$.

For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 11 & 12 & 13 & 14 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

then

$$B = \begin{bmatrix} 1 & 2 \\ 11 & 12 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 1 \end{bmatrix}$$

are submatrices of $A$. 
Matrix arithmetic

Scalar multiplication: If $A = [a_{ij}]$ is an $m \times n$ matrix and $\alpha$ is a scalar (a real or complex number), then $C = \alpha A$ is the matrix with entries $c_{ij} = \alpha a_{ij}$. For example

\[3 \cdot \begin{bmatrix} 1 & 2 & 3 & 4 \\ 11 & 12 & 13 & 14 \\ 4 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 33 & 36 & 39 & 42 \\ 12 & 9 & 6 & 3 \end{bmatrix}\]

and

\[\left(-\frac{1}{2}\right) \cdot \begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3/2 \\ -5/2 & -2 \end{bmatrix}\]
**Addition:** If \( A = [a_{ij}] \) and \( B = [b_{ij}] \) are two matrices of the *same size*, then \( C = A + B \) is the matrix with entries \( c_{ij} = a_{ij} + b_{ij} \). If the matrices \( A \) and \( B \) do not have the same (exact) size, then \( A + B \) is *not defined*. For example,

\[
\begin{bmatrix}
1 & 2 \\
11 & 12
\end{bmatrix} + \begin{bmatrix}
1 & -1 \\
-3 & 2
\end{bmatrix} = \begin{bmatrix}
2 & 1 \\
8 & 14
\end{bmatrix}
\]

but

\[
\begin{bmatrix}
1 & 2 & 3 \\
11 & 12 & 5
\end{bmatrix} + \begin{bmatrix}
1 & -1 \\
-3 & 2
\end{bmatrix}
\]

is simply *not defined.*

If \( A = [a_{ij}] \) is an \( m \times n \) matrix, then \( -A = [-a_{ij}] \). Observe that

\[
A + (-A) = O,
\]

the \( m \times n \) *zero matrix*, all of whose entries are \( = 0 \).
Properties of addition and scalar multiplication.

For $A$, $B$ and $C$ any matrices in $\mathbb{R}^{m\times n}$ and for $O$ the $m \times n$ zero matrix, the following properties hold:

I. $A + B = B + A$

II. $A + (B + C) = (A + B) + C$

III. $A + O = A$

IV. $A + (-A) = O$

V. $1 \cdot A = A$

VI. $\alpha(\beta A) = (\alpha \beta)A$

VII. $\alpha(A + B) = \alpha A + \alpha B$

VIII. $(\alpha + \beta)A = \alpha A + \beta A$

Comments:

1. These properties generalize, and follow from the same properties for addition and (scalar) multiplication of real (or complex) numbers.

2. The fact that these properties hold make $\mathbb{R}^{m\times n}$ a vector space over $\mathbb{R}$. Of particular importance, is the fact that $\mathbb{R}^n$ (the space of column/row vectors) is a vector space.
Linear combinations:

If \( \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k \) are vectors in \( \mathbb{R}^n \) and \( c_1, c_2, \ldots, c_k \) are scalars then the vector

\[
y = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_k \mathbf{b}_k
\]

is a \textit{linear combination} of the vectors \( \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k \), with coefficients \( c_1, c_2, \ldots, c_k \). Note that \( y \) is also in \( \mathbb{R}^n \).

\textbf{Example.} The linear combination of \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
-1 \\
2 \\
-4
\end{bmatrix}
\]

and \[
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}
\]

with coefficients 3, 2 and 1 is

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = 3 \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} + 2 \begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix} + 1 \begin{bmatrix}
-4 \\
1 \\
-4
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]
Matrix multiplication:

I. Multiplying a column vector by a row vector: If \( \mathbf{r} = [r_1 \ldots r_k] \) is a row (vector) with \( k \) entries and \( \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \) is a column (vector), also with \( k \) entries, then \( \mathbf{r} \cdot \mathbf{c} \) is defined by

\[
\mathbf{r} \cdot \mathbf{c} = [r_1 \ldots r_k] \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = r_1 c_1 + \cdots + r_k c_k
\]

Comments:

1. This product is \textit{not} commutative. I.e., \( \mathbf{r} \cdot \mathbf{c} \neq \mathbf{c} \cdot \mathbf{r} \).

2. If \( \mathbf{r} \) and \( \mathbf{c} \) have different numbers of entries, then the product is \textit{not} defined.
Example:

\[
\begin{bmatrix}
1 & 2 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
2 \\
0 \\
3 \\
1
\end{bmatrix}
= 1 \cdot 2 + 2 \cdot 0 + (-1) \cdot 3 + 3 \cdot 1 = 2.
\]

II. Multiplying a column by a matrix: If \( A \) is an \( m \times n \) matrix and \( \mathbf{b} \) is a column vector with \( n \) entries, then \( A \cdot \mathbf{b} \) is the column vector in \( \mathbb{R}^m \) whose entries are the row-column products of the rows of \( A \) with the column \( \mathbf{b} \):

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{bmatrix}
= \begin{bmatrix}
a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\
a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\
\vdots \\
a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n
\end{bmatrix}
\]
Example:

\[
\begin{bmatrix}
1 & 2 & -1 & 3 \\
2 & 0 & 4 & -3 \\
3 & -1 & 3 & -2
\end{bmatrix}
\begin{bmatrix}
2 \\
0 \\
3 \\
1
\end{bmatrix}
= \begin{bmatrix}
1 \cdot 2 + 2 \cdot 0 + (-1) \cdot 3 + 3 \cdot 1 \\
2 \cdot 2 + 0 \cdot 0 + 4 \cdot 3 + (-3) \cdot 1 \\
3 \cdot 2 + (-1) \cdot 0 + 3 \cdot 3 + (-2) \cdot 1
\end{bmatrix}
= \begin{bmatrix}
2 \\
13 \\
13
\end{bmatrix}
\]
III. Matrix multiplication, in general: If $A$ is an $m \times k$ matrix with rows $a_1, \ldots, a_m$ and $B$ is a $k \times n$ matrix with columns $b_1, \ldots, b_n$, then the product $AB$ is defined as

$$AB = \begin{bmatrix}
  a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_j & \cdots & a_1 \cdot b_n \\
  a_2 \cdot b_1 & a_2 \cdot b_2 & \cdots & a_2 \cdot b_j & \cdots & a_1 \cdot b_n \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_i \cdot b_1 & a_i \cdot b_2 & \cdots & a_i \cdot b_j & \cdots & a_i \cdot b_n \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_m \cdot b_1 & a_m \cdot b_2 & \cdots & a_m \cdot b_j & \cdots & a_m \cdot b_n
\end{bmatrix}$$

In words: the $(i, j)^{th}$ entry in the product $AB$ is the row-column product of the $i^{th}$ row of $A$ with the $j^{th}$ column of $B$.

Important: For the product to be defined, each row of $A$ must have the same number of entries as each column of $B$. 
Observations:

1. The $j^{\text{th}}$ column of the product $AB$ is $A b_j$, the product of the $j^{\text{th}}$ column of $B$ by the matrix $A$.

2. The number of columns in $AB$ is equal to the number of columns in $B$. The number of rows in $AB$ is equal to the number of rows in $A$.

3. If $A$ is $m \times k$ and $B$ is $k \times n$, then $AB$ is $m \times n$.

4. If the number of columns in $A$ is not equal to the number of rows in $B$, then the product $AB$ is not defined.
Example:

\[
\begin{bmatrix}
1 & 1 & 0 \\
2 & 1 & 3 \\
4 & -1 & 1 \\
0 & 3 & -2 \\
\end{bmatrix}
\begin{bmatrix}
1 & -1 & 1 & 2 \\
2 & 1 & -3 & 0 \\
1 & 4 & -1 & 1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 + 2 + 0 & -1 + 1 + 0 & 1 - 3 + 0 & 2 + 0 + 0 \\
2 + 2 + 3 & -2 + 1 + 12 & 2 - 3 - 3 & 4 + 0 + 3 \\
4 - 2 + 1 & -4 - 1 + 4 & 4 + 3 - 1 & 8 + 0 + 1 \\
0 + 6 - 2 & 0 + 3 - 8 & 0 - 9 + 2 & 0 + 0 - 2 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
3 & 0 & -2 & 2 \\
7 & 11 & -6 & 7 \\
3 & -1 & 6 & 9 \\
4 & -5 & -7 & -2 \\
\end{bmatrix}
\]
Properties of matrix multiplication:

1. Matrix multiplication **distributes over sums**: If $A$ is an $m \times n$ matrix and $B$ and $C$ are both $n \times k$ matrices then
   \[
   A(B + C) = AB + AC
   \]
   Likewise, if $D$ and $E$ are both $l \times m$ matrices and $F$ is an $m \times n$ matrix, then
   \[
   (D + E)F = DF + EF.
   \]

2. Matrix multiplication is **associative**: If $A$ is an $m \times n$ matrix, $B$ is an $n \times p$ matrix and $C$ is a $p \times q$ matrix, then
   \[
   (AB)C = A(BC).
   \]

3. Matrix multiplication **commutes with scalar multiplication**: If $A$ is $m \times n$, $B$ is $n \times l$ and $\alpha$ is a scalar, then
   \[
   A(\alpha B) = \alpha(AB) = (\alpha A)B.
   \]
4. For every positive integer $n$, there is an $n \times n$ identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

such that $I_n A = A$ for every $n \times m$ matrix $A$ and $B I_n = B$ for every $k \times n$ matrix $B$.

**Remark:** The identity matrix is unique. I.e., if there is another matrix $J$ such that $J A = A J = A$ for all $n \times n$ matrices $A$, then $J = I_n$. Because

$$J = J I_n = I_n$$
**Properties that matrix multiplication does not have:**

1. Matrix multiplication is **not commutative:**

(i) If $A$ is $m \times n$ and $B$ is $n \times k$, then $AB$ is defined, but $BA$ is not defined if $k \neq m$.

(ii) In the case that $A$ is $m \times n$ and $B$ is $n \times m$, then both $AB$ and $BA$ are defined, but if $n \neq m$, then $AB$ and $BA$ have **different sizes** and so cannot be equal.

(iii) Even in the case that both $A$ and $B$ are $n \times n$ (square) matrices the products $AB$ and $BA$ are usually **not equal** ...

\[
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]
2. If $AB = O$ (the zero matrix), it does not follow that $A = O$ or $B = O$. E.g.,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. **Cancellation** does not hold in general. I.e., If $AB = AC$, it does not generally follow that $B = C$. E.g.,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$$
The **transpose of a matrix**: If $A$ is an $m \times n$ matrix with entries $a_{ij}$, then $A^T$ is the $n \times m$ matrix with entries $\alpha_{st}$ satisfying

$$\alpha_{st} = a_{ts}.$$ 

Loosely speaking, the $j^{\text{th}}$ column of $A^T$ is equal to the $j^{\text{th}}$ row of $A$, rotated by $90^\circ$.

**Example:**

$$\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
0 & 0 & 0 & 0
\end{bmatrix}^T = \begin{bmatrix}
1 & 5 & 0 \\
2 & 6 & 0 \\
3 & 7 & 0 \\
4 & 8 & 0
\end{bmatrix}$$
Transposes and arithmetic:

1. The transpose of a sum is the sum of the transposes:

\[(A + B)^T = A^T + B^T\]

2. The transpose of a scalar multiple is the scalar multiple of the transpose:

\[(\alpha \cdot A)^T = \alpha \cdot A^T\]

3. The transpose of a matrix product is the product of the transposes \textit{in reverse order}:

\[(AB)^T = B^T A^T\]
Example:

\[
\begin{pmatrix}
1 & 1 & 2 \\
2 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 & -1 \\
1 & 0 \\
1 & 1
\end{pmatrix}
= 
\begin{pmatrix}
5 & 1 \\
5 & -1
\end{pmatrix}
\]

\[
\begin{pmatrix}
5 & 5 \\
1 & -1
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & 1 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
1 & 0 \\
2 & 1
\end{pmatrix}
= 
\begin{pmatrix}
2 & -1 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 2 \\
2 & 0 & 1
\end{pmatrix}
\]
The matrix form of a linear system.

Consider the linear system

\[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \]
\[ \vdots \]
\[ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \]
\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \]

The \( j \)th equation can be written as

\[
\begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_j
\]
and the entire system can expressed as a single *matrix equation*:

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  \vdots  & \vdots  & \vdots  & \ddots & \vdots  \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
= 
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix}
\]

In other words, If \( S \) is a linear system of \( m \) equations in \( n \) variables, then this system can be expressed in the form \( A\mathbf{x} = \mathbf{b} \) where \( A \) is the \( m \times n \) coefficient matrix of the system, \( \mathbf{b} \) is the \( m \times 1 \) vector of constant terms and \( \mathbf{x} \) is the \( n \times 1 \) vector of variables.

**Observation:** The set of solutions of a linear system of \( m \) equations in \( n \) variables can be thought of as a set of vectors in \( \mathbb{R}^n \).
**Example:** The linear system

\[
\begin{align*}
2x_1 + x_2 - x_3 &= 1 \\
x_1 - 2x_2 + 3x_3 &= -1 \\
3x_1 - x_2 + 4x_3 &= 0
\end{align*}
\]  

(1)

can be expressed as the matrix equation

\[
\begin{bmatrix}
2 & 1 & -1 \\
1 & -2 & 3 \\
3 & -1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix}
\]  

(2)

And the set of solutions of the linear system (1) can be expressed as the set of vectors \( \mathbf{x} \) in \( \mathbb{R}^3 \) that satisfy the equation (2).
Elementary row operations on the augmented matrix of (1) gives

\[
\begin{bmatrix}
2 & 1 & -1 & 1 \\
1 & -2 & 3 & -1 \\
3 & -1 & 4 & 0
\end{bmatrix} \Rightarrow \text{EROs} \Rightarrow
\begin{bmatrix}
1 & 0 & 5 & \frac{1}{5} \\
0 & 1 & 1 & \frac{3}{5} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

So, in parametric form with parameter \( t \), the set of solutions of (1) are given by

\[
x_1 = \frac{1}{5} - 5t, \quad x_2 = \frac{3}{5} - t, \quad x_3 = t.
\]

As a set of vectors in \( \mathbb{R}^3 \), the set of solutions is the set of vectors:

\[
\left\{ \begin{bmatrix}
\frac{1}{5} - 5t \\
\frac{3}{5} - t \\
t
\end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix}
\frac{1}{5} \\
\frac{3}{5} \\
0
\end{bmatrix} + t \begin{bmatrix}
-5 \\
-1 \\
1
\end{bmatrix} : t \in \mathbb{R} \right\}
\]