**Invertible matrices.**

An $n \times n$ matrix $A$ is *invertible* if there is an $n \times n$ matrix $B$ such that

$$AB = BA = I_n.$$ 

In this case, we write $B = A^{-1}$.

If $A^{-1}$ does not exist, then $A$ is said to be *noninvertible* or *singular*.

**Example:**

\[
\begin{bmatrix}
3 & 2 \\
4 & 3
\end{bmatrix}
\begin{bmatrix}
3 & -2 \\
-4 & 3
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
3 & -2 \\
-4 & 3
\end{bmatrix}
\begin{bmatrix}
3 & 2 \\
4 & 3
\end{bmatrix}
\]

so

\[
\begin{bmatrix}
3 & 2 \\
4 & 3
\end{bmatrix}^{-1}
= 
\begin{bmatrix}
3 & -2 \\
-4 & 3
\end{bmatrix}
**Observation:** If $A^{-1} = B$, then $AB = BA = I_n$, but this also means that $B^{-1} = A$. In other words (and not surprisingly) $(A^{-1})^{-1} = A$.

**Fact:** The inverse of an invertible matrix is **unique**.

Suppose that $A$ is invertible and $A^{-1} = B$, and further suppose that $AC = I_n$. Then

$$B = BI_n = B(AC) = (BA)C = I_n C = C.$$ 

**Right and left inverses.**

If $A$ is an $n \times n$ matrix and $B$ is an $n \times n$ matrix satisfying $AB = I_n$, then $B$ is called a *right inverse* of $A$. Likewise, if $C$ is an $n \times n$ matrix such that $CA = I_n$, then $C$ is called a *left inverse* of $A$.

**Fact:** If $A$ has a right inverse $B$ and a left inverse $C$, then $B = C$ so $A$ is invertible.

$$B = BI_n = B(AC) = (BA)C = I_n C = C.$$
**Fact:** If $A$ has a unique right inverse $B$, then $B$ is also a left inverse of $A$, so $A^{-1} = B$ (and $A$ is invertible).

Suppose that $B$ is the unique right inverse of $A$, then

$$A(BA + B - I_n) = A(BA) + AB - A = (AB)A + AB - A = A + I_n - A = I_n,$$

so $(BA + B - I_n)$ is a right inverse of $A$. But $B$ is the unique right inverse of $A$, so

$$BA + B - I_n = B \implies BA - I_n = O \implies BA = I_n$$

so $B$ is a left inverse, as promised.

**Conclusion:** To find the inverse of a square matrix $A$, it is enough to find a right inverse of that matrix and show that it is unique.

This task reduces to solving a system of $n$ linear systems ...
A matrix is determined by its columns, so finding a right inverse \( X \) for an \( n \times n \) matrix \( A \) is the same as finding \( n \) column vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) in \( \mathbb{R}^n \) that are the columns of \( X \), i.e., vectors for which

\[
A \cdot [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n] = I_n.
\]

On the other hand,

\[
A \cdot [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n] = I_n \Rightarrow [A\mathbf{x}_1 \ A\mathbf{x}_2 \ \cdots \ A\mathbf{x}_n] = I_n
\]

\[
\Rightarrow [A\mathbf{x}_1 \ A\mathbf{x}_2 \ \cdots \ A\mathbf{x}_n] = [\mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_n],
\]

where \( \mathbf{e}_j \) is the \( j^{\text{th}} \) column of \( I_n \).

In other words, to find a right inverse for \( A \), we have to solve the \( n \) linear systems

\[
A\mathbf{x}_1 = \mathbf{e}_1, \ A\mathbf{x}_2 = \mathbf{e}_2, \ldots, \ A\mathbf{x}_j = \mathbf{e}_j, \ldots, \ A\mathbf{x}_n = \mathbf{e}_n \ldots
\]
... where

\[ e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{row } j \]

To solve any one of these linear systems, the \( j^{\text{th}} \) system for example, we use Gauss-Jordan elimination:

\[
\begin{bmatrix} A | e_j \end{bmatrix} \Rightarrow \text{EROs} \Rightarrow \begin{bmatrix} A^* | x_j \end{bmatrix}
\]

where \( A^* \) is the reduced echelon form of \( A \).
Observation: If rank\( (A) = n \), then \( A^* = I_n \) and the system \( Ax_j = e_j \) has a unique solution for each \( e_j \). Why? 

*Because there is a *pivot in every column* of the coefficient matrix \( A \), common to all these systems.*

Conclusion: If rank\( (A) = n \), there is a unique \( n \times n \) matrix \( X = [x_1 \ x_2 \ldots \ x_n] \) satisfying \( AX = I_n \). I.e., \( A \) has a unique right inverse, so \( A \) is invertible.

On the other hand: if \( A \) is invertible and \( b \) is any vector in \( \mathbb{R}^n \), then the equation \( Ax = b \) has the *unique* solution \( x = A^{-1}b \). But this means that the linear system with coefficient matrix \( A \) (and vector of constant terms \( b \)) has a unique solution, so rank\( (A) = n \).

Grand conclusion: the \( n \times n \) matrix \( A \) is invertible if and only if rank\( (A) = n \).
The relationship between the invertibility of $A$ and the linear systems $Ax_j = e_j$ also provides a practical way of finding $A^{-1}$.

**Key observation:** The EROs used to reduce $[A|e_1]$ to $[A^*|x_1]$ are exactly the same EROs used to reduce $[A|e_j]$ to $[A^*|x_j]$, for each and every $j$.

Therefore, to find $A^{-1}$ (if it exists) we apply Gauss-Jordan elimination to the $n \times 2n$ matrix $[A|I_n]$:

$$[A|I_n] \Rightarrow \text{EROs} \Rightarrow [A^*|X]$$

where (as before) $A^*$ is the reduced echelon form of $A$.

(*) If $A^* = I_n$, then $A$ is invertible and $X = A^{-1}$.

(*) If $A^* \neq I_n$, then $A$ is not invertible.

(In the second case, we will usually see that $\text{rank}(A) < n$, before we complete the elimination process.)
Example: Find the inverse of

\[
A = \begin{bmatrix}
1 & 1 & 3 \\
2 & -1 & 1 \\
2 & 2 & -1
\end{bmatrix},
\]

if it exists or show that \(A\) is singular.

Reducing \([A|I_n]\) gives...

\[
\begin{bmatrix}
1 & 1 & 3 & 1 & 0 & 0 \\
2 & -1 & 1 & 0 & 1 & 0 \\
2 & 2 & -1 & 0 & 0 & 1
\end{bmatrix} \Rightarrow \text{EROs} \Rightarrow
\begin{bmatrix}
1 & 0 & 0 & -\frac{1}{21} & \frac{1}{3} & \frac{4}{21} \\
0 & 1 & 0 & \frac{4}{21} & -\frac{1}{3} & \frac{5}{21} \\
0 & 0 & 1 & \frac{2}{7} & 0 & -\frac{1}{7}
\end{bmatrix}
\]

(as you should verify)
so $A$ is invertible and

$$A^{-1} = \begin{bmatrix} -\frac{1}{21} & \frac{1}{3} & \frac{4}{21} \\ \frac{4}{21} & -\frac{1}{3} & \frac{5}{21} \\ \frac{2}{7} & 0 & -\frac{1}{7} \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{21} & \frac{1}{3} & \frac{4}{21} \\ \frac{4}{21} & -\frac{1}{3} & \frac{5}{21} \\ \frac{2}{7} & 0 & -\frac{1}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$