Review Questions for Final Exam

The final exam will be comprehensive, with an emphasis on the second half of the quarter. The questions below come from the second half of the course, and to be fully prepared, you should also study the midterm review questions (and the quizzes and the homework).

In addition to computational problems, you may be tested on concepts and definitions by means of True/False questions and/or short proofs. In particular, I recommend reviewing the True/False questions from the homework of the sections that we covered.

1. Consider the subspace \( W = \text{span} \left( \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right) \) of \( \mathbb{R}^3 \).

   (i) Find an orthogonal basis for \( W \).

   (ii) Find the vector in \( W \) that is closest to the vector \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

   (iii) Can you solve (ii) without solving (i)? If so, how?

2. Is the matrix \( A \) below diagonalizable? If so, find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \). If not, explain why not.

   \[
   A = \begin{bmatrix}
   2 & 1 & 1 \\
   -1 & 1 & 3 \\
   1 & 0 & -2
   \end{bmatrix}.
   \]

   Hint: Before trying to solve a cubic equation (why would you do that?), see problem ??.

3. In a large forest, foxes prey on rabbits while the rabbits feed on the (unlimited) vegetation. The evolution of the fox and rabbit populations in the forest is modeled by the following linear system:

   \[
   \begin{bmatrix}
   F_{k+1} \\
   R_{k+1}
   \end{bmatrix} = \begin{bmatrix}
   0.5 & 0.3 \\
   -p & 1.2
   \end{bmatrix} \cdot \begin{bmatrix}
   F_k \\
   R_k
   \end{bmatrix},
   \]

   where \( F_k \) is the size of the fox population in year \( k \), \( R_k \) is the size of the rabbit population in year \( k \) and \( p \) is a positive number called the predation parameter, that accounts for deaths in the rabbit population due to predation by foxes. The matrix, \( T_p \), on the right hand side of the equation is called the transition matrix of the model.

   (i) Find the eigenvalues and corresponding eigenvectors for the transition matrix when the predation parameter is \( p = 0.275 \).

   (ii) Find approximate values for \( F_{20} \) and \( R_{20} \), given that \( F_0 = 4 \), \( R_0 = 20 \) and \( p = 0.275 \). Explain your work.
(iii) With $F_0, R_0$ and $p$ as in part b., what can you say about the ratio $R_k/F_k$ as $k$ grows larger?

(iv) With $p = 0.275$, find the critical ratio $\rho^*$ such that if $R_0/F_0 > \rho^*$, then both populations survive, and if $R_0/F_0 \leq \rho^*$, then both populations die off. Explain your work.

**Hint for (iv):** Cramer's rule is useful here.

4. Find bases for the (i) null space, (ii) column space and (iii) row space of the matrix

\[
A = \begin{bmatrix}
1 & 1 & 3 & 3 & 1 \\
1 & 2 & 5 & 4 & 0 \\
2 & 0 & 2 & 4 & 1 \\
2 & 1 & 4 & 5 & 1
\end{bmatrix}
\]

5. Let $M$ be an $n \times n$ matrix with real entries such that the sum of the entries in each column of $M$ is equal to the same value $\kappa$. Show that $\kappa$ is an eigenvalue of $M$.

**Hint:** What is the sum of the rows of the matrix $M - \kappa I$? What does this imply?

6. Consider the set of vectors

\[
\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

(i) Show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

(ii) Find $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}}$, the coordinate vector of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B}$.

7. Suppose that the $n \times n$ matrix $M$ has $n$ distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that $\det M = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.

8. Compute the determinants

\[
\det \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} = \quad \det \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix} = \quad \det \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} =
\]

9. Find the inverses of the three matrices in the problem above.

10. Consider the matrix

\[
A = \begin{bmatrix}
0.3 & 0.8 \\
0.7 & 0.2
\end{bmatrix}
\]

Show that for any vector $\vec{u}$ in $\mathbb{R}^2$, there is a vector $\vec{u}_0$ (that depends on $\vec{u}$) such that

\[
\lim_{k \to \infty} A^k \vec{u} = \vec{u}_0.
\]
11. Let $H$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$ 

Find a basis for $H^\perp$, the orthogonal complement of $H$.

12. Find the least-squares (approximate) solution of the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. $$