Solutions

1. The degree 10 Taylor polynomial for \( f(x) = e^x \) centered at \( x_0 = 0 \) is given by

\[
T_{10}(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \cdots + \frac{f^{(10)}(0)}{10!}(x - 0)^{10}
\]

\[
= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800},
\]

since

\[
f^{(k)}(x) = \frac{d^k}{dx^k}(e^x) = e^x
\]

for all \( k \), and \( e^0 = 1 \).

Comment: If we use \( T_{10}(1) \) to approximate the value of \( e = e^1 \), then we obtain the estimate

\[
e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880} + \frac{1}{3628800}
\]

\[
= 2.7182818011463845.
\]

The first 7 digits after the decimal point are correct.

2. If \( g(x) = \sqrt[3]{x} = x^{1/3} \), then \( g'(x) = x^{-2/3}/3 \) and \( g''(x) = -2x^{-5/3}/9 \). Evaluating \( g(x) \) and its first two derivatives at \( x = 1000 \), we find that

\[
g(1000) = 10, \quad g'(1000) = \frac{1}{300} \quad \text{and} \quad g''(1000) = -\frac{2}{900000} = -\frac{1}{450000}.
\]

The second degree Taylor polynomial for \( g(x) = \sqrt[3]{x} \), centered at \( x = 1000 \) is therefore equal to

\[
T_2(x) = 10 + \frac{(x - 1000)}{300} - \frac{(x - 1000)^2}{900000}.
\]

Using this polynomial we obtain the approximation

\[
\sqrt[3]{1001} \approx 10 + \frac{1}{300} - \frac{1}{900000} = 10.003332222\ldots,
\]

(the decimal expansion has nothing but 2’s from the seventh point on).

The first 9 digits after the decimal point are correct here.

3. First, we compute the derivatives up to and including order 6 of \( f(x) = \ln x \):

\[
f'(x) = x^{-1}, \quad f''(x) = -x^{-2}, \quad f'''(x) = 2x^{-3}, \quad f^{(4)}(x) = -6x^{-4},
\]
\[ f^{(5)}(x) = 24x^{-5} \quad \text{and} \quad f^{(6)}(x) = -120x^{-6}. \]

Evaluating \( \ln x \) and its derivatives at \( x = 1 \), we find that

\[
T_6(x) = \ln 1 + 1^{-1} \cdot (x - 1) + \frac{-1^{-2}}{2} \cdot (x - 1)^2 + \frac{2 \cdot 1^{-3}}{6} (x - 1)^3 + \frac{-6 \cdot 1^{-4}}{24} (x - 1)^4
+ \frac{24 \cdot 1^{-5}}{120} (x - 1)^5 + \frac{-120 \cdot 1^{-6}}{720} (x - 1)^6
= (x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 - \frac{1}{4} (x - 1)^4 + \frac{1}{5} (x - 1)^5 - \frac{1}{6} (x - 1)^6.
\]

The graphs of \( y = \ln x \) (the black line) and \( y = T_6(x) \) (the dashed red line) appear above.

Next, we evaluate:

\[
T_6(0.8) = (0.8 - 1) - \frac{(0.8 - 1)^2}{2} + \frac{(0.8 - 1)^3}{3} - \frac{(0.8 - 1)^4}{4} + \frac{(0.8 - 1)^5}{5} - \frac{(0.8 - 1)^6}{6}
= -\frac{1}{5} - \frac{1}{50} - \frac{1}{375} - \frac{1}{2500} - \frac{1}{15625} - \frac{1}{93750}
= -\frac{41389}{187500}.
\]

I.e., \( \ln 0.8 \approx -\frac{41389}{187500} \) and \( \ln 1.25 = -\ln 0.8 \approx \frac{41389}{187500} \) (since \( 1.25 = 1/0.8 \)).

(Note that \( -\frac{41389}{187500} = -0.2231413333 \). The first five decimal digits of this number are the same as the first five decimal digits of \( \ln 0.8 \).)
4. We have \( f'(x) = \frac{1}{2}x^{-1/2} \), \( f''(x) = -\frac{1}{3}x^{-3/2} \) and \( f'''(x) = \frac{3}{8}x^{-5/2} \), so

\[
 f(100) = 10, \quad f'(100) = \frac{1}{20}, \quad f''(100) = -\frac{1}{4000} \quad \text{and} \quad f'''(100) = \frac{3}{800000}.
\]

Thus, the degree 3 Taylor polynomial for \( f(x) = \sqrt{x} \), centered at \( x_0 = 100 \) is

\[
 T_3(x) = 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2 + \frac{1}{1600000}(x - 100)^3.
\]

From this we obtain the estimate

\[
 \sqrt{110} \approx T_3(110) = 10 + \frac{1}{2} - \frac{1}{80} + \frac{1}{1600} = 10.488125.
\]

(The first three decimal digits are correct here.)