1. Denote by \( X \) the vertical dimension and by \( Y \) the horizontal dimension of Farmer Jones’ rectangular plot, (see Figure 1).

\[
\begin{array}{c|c|c|c}
X & & & Y \\
\end{array}
\]

Figure 1: Farmer Jones’ vegetable garden.

Then, the cost of the fencing is

\[ C(X,Y) = 8 \cdot (4X) + 12 \cdot (2X + 2Y) = 56X + 24Y. \]

Since \( XY = 4800 \), we can substitute \( Y = 4800/X \) into the expression above to obtain the cost as a function of \( X \) alone:

\[ C(X) = 56X + \frac{115200}{X}. \]

We want to find the absolute minimum of this function on the interval \((0, \infty)\) (since lengths must be positive).

(a) Critical points:

\[
\frac{dC}{dX} = 56 - \frac{115200}{X^2} = 0 \quad \Rightarrow \quad X = \pm \frac{120}{\sqrt{7}}.
\]

So there is only one critical point in \((0, \infty)\): \( X_0 = \frac{120}{\sqrt{7}} \approx 45.36 \).

(b) Analysis:

\[
\frac{d^2C}{dX^2} = \frac{230400}{X^3} \quad \Rightarrow \quad \frac{d^2C}{dX^2}\bigg|_{X=\frac{120}{\sqrt{7}}} > 0.
\]

Thus, by the second derivative test, \( C(120/\sqrt{7}) \approx 5079.84 \) is a relative minimum value, and since \( X^* = 120/\sqrt{7} \) is the only critical point in \((0, \infty)\), this is the absolute minimum cost. This minimum is attained when \( X = X^* \approx 45.36 \) and \( Y = 4800/X^* \approx 105.83 \), (so the figure above is not to scale).

(c) Conclusion: The minimum cost to Farmer Jones is

\[ C^* = C(X^*) = 56X^* + \frac{115200}{X^*} \approx 5079.84. \]
2. The firm’s profit function is \( \Pi = r - c = pq - (0.1q^2 + 10q + 1500) \). Solving the demand equation for \( p \) (in terms of \( q \)) gives \( p = 80 - 0.25q \), so the profit function is

\[
\Pi(q) = -0.35q^2 + 70q - 1500.
\]

(a) Critical point:

\[
\frac{d\Pi}{dq} = -0.7q + 70 = 0 \implies q^* = 100.
\]

(b) Analysis and conclusion:

\[
\frac{d^2\Pi}{dq^2} = -0.7 < 0 \text{ for all } q.
\]

So \( \Pi^* = \Pi(q^*) = \Pi(100) = 2000 \) is the absolute maximum profit for the firm because \( \Pi^* \) is a relative minimum by the second derivative test and \( q^* \) is the only critical point. The profit maximizing price is \( p^* = 80 - 0.25q^* = 55 \).

3. (a) We repeat the profit maximizing exercise above, but with a new cost function. The new cost function is \( c_1(q) = c(q) + 2.8q = -0.25q^2 + 12.8q + 1500 \), so the new profit function is

\[
\Pi_1 = -0.35q^2 + 67.2q + 1500.
\]

The new profit-maximizing level of output is found by finding the critical point(s) of \( \Pi_1 \):

\[
\frac{d\Pi_1}{dq} = -0.7q + 67.2 = 0 \implies q_1 = 96.
\]

This still yields the absolute maximum profit, since \( \frac{d^2\Pi_1}{dq^2} = -0.7 < 0 \) and correspondingly a new profit-maximizing price \( p_1 = 80 - 0.25q_1 = 56 \). Thus, the firm raises the price of its product by $1.

(b) The firm’s new (maximum) profit is \( \Pi_1(96) = $1725.60 \).

(c) The government’s tax revenue will be \( 2.80 \cdot 96 = $268.80 \).

(d) Consumers (effectively) pay for \( 1 \cdot 96 = $96 \) of the government’s tax revenue.

Comment: If the government imposes a fixed fee of $270.00 on the firm, instead of the per-unit tax, the firm’s profit-maximizing price doesn’t change (so consumers are happier); the firm’s new profit is $1730.00 (so the firm is happier); and the government’s revenue is $270.00 (so the government is happier).

4. (a) The cost to the firm of using \( k \) units of capital and \( l \) units of labor is 

\[
1000k + 1250l,
\]

and since the firm’s production budget is 100000, this imposes a constraint on the levels of capital and labor input that the firm can use, namely

\[
1000k + 1250l = 100000.
\]
This allows us to express the firm’s output as a function of \( l \) alone (or \( k \) alone, if you prefer):

\[
q = 10k^{1/4}l^{3/4} = 10 \left( 100 - 1.25l \right)^{1/4} l^{3/4}.
\]

Next, we observe that this is a closed interval optimization problem, since \( l \geq 0 \) and \( k \geq 0 \), implies that \( 0 \leq l \leq 80 \) (since \( 100 - 1.25l \geq 0 \) implies \( l \leq 80 \)).

i. Critical point(s):

\[
\frac{dq}{dl} = 10 \left[ \frac{1}{4} \cdot (100 - 1.25l)^{-3/4} \cdot (-1.25) \cdot l^{3/4} + (100 - 1.25l)^{1/4} \cdot \frac{3}{4} l^{-1/4} \right]
\]

\[
= 10 \left[ \frac{3(100 - 1.25l)^{1/4}}{4l^{1/4}} - \frac{5l^{3/4}}{16(100 - 1.25l)^{3/4}} \right]
\]

\[
= 10 \left[ \frac{12(100 - 1.25l) - 5l}{16^{1/4}(100 - 1.25l)^{3/4}} \right] = \frac{12000 - 200l}{16^{1/4}(100 - 1.25l)^{3/4}}.
\]

Solving \( dq/dl = 0 \) means \( 12000 - 200l = 0 \), giving one critical point, \( l_0 = 60 \). We also note that \( dq/dl \) is undefined when \( l = 0 \) and \( l = 80 \), so the other two critical points coincide with the endpoints of the interval.

ii. Evaluate: \( q(0) = q(80) = 0 \) and \( q(60) \approx 482.057 \), so output is maximized when \( l = 60 \) and \( k = 100 - 1.25 \cdot 60 = 25 \).

(b) In this case, we want to find the minimum value of the cost \( c = 1000k + 1250l \), and we use the constraint \( 10k^{1/4}l^{3/4} = 2500 \) to express this as a function of \( l \) alone:

\[
10k^{1/4}l^{3/4} = 2500 \quad \Rightarrow \quad k = 250^4l^{-3} \quad \Rightarrow \quad c = 1000 \cdot (250)^4l^{-3} + 1250l.
\]

The interval in this case is \((0, \infty)\).

i. Critical point(s):

\[
\frac{dc}{dl} = -3000(250)^4l^{-4} + 1250 = 0 \quad \Rightarrow \quad l^4 = \frac{3000(250)^4}{1250} \quad \Rightarrow \quad l = \pm(250)^4 \sqrt[4]{\frac{3000}{1250}}.
\]

so there is exactly one critical point, \( l_0 = (250)^4 \sqrt[4]{\frac{3000}{1250}} \) in the interval \((0, \infty)\).

ii. Analysis: \( \frac{d^2c}{dl^2} = 12000(250)^4l^{-5} > 0 \) for all \( l > 0 \), so the cost function \( c(l) \) is concave up on the interval \((0, \infty)\) and \( c(l_0) \approx 518610.81 \) is the firm’s absolute minimum cost for producing 2500 units. This cost is achieved for \( l_0 \approx 311.166 \) and \( k_0 \approx 129.653 \).

(c) To answer this question, we replace “\((250)\)” by “\((q/10)\)” and repeat, word for word, the work of the previous part. In particular, this gives the critical point

\[
l_q = (q/10) \cdot \sqrt[4]{\frac{3000}{1250}} \approx 0.1245q
\]
and the minimal cost (of producing $q$ units)

$$c(q) = 1000(q/10)^4(0.1245q)^{-3} + 1250 \cdot 0.1245q \approx 207.444q.$$ 

5. The average cost function is $\bar{c} = \frac{c}{q} = 0.02q + 20 + \frac{800}{q}$. The interval in this case is $(0, \infty)$, because output $q$ must be positive.

(a) Critical point(s):

$$\frac{d\bar{c}}{dq} = 0.02 - \frac{800}{q^2} = 0 \implies q = \pm \sqrt{\frac{800}{0.02}} = \pm 200,$$

so there is only one critical point, $q_0 = 200$, in the interval $(0, \infty)$.

(b) Analysis:

$$\frac{d^2\bar{c}}{dq^2} = \frac{1600}{q^3} > 0 \quad \text{for all } q > 0,$$

so $\bar{c}(200) = 28$ is the firm’s absolute minimum average cost, since the average cost function is concave up on $(0, \infty)$. 