Review Questions 1

Solutions

1. Compute the differentials of the functions below.
   a. \( y = x^2 - 3x + 1 \), \( dy = (2x - 3) \, dx \)
   b. \( u = e^{x^2 - 3x + 1} \), \( du = e^{x^2 - 3x + 1} \,(2x - 3) \, du \)

2. Use differentials to estimate \( \sqrt[3]{28} \). Express your answer as a simple fraction, \( a/b \), not in decimal form.

   We use the approximation formula \( f(x_0 + dx) \approx f(x_0) + dy \). First, we identify the function, which is straightforward: \( f(x) = x^{1/3} \). Next, we identify \( x_0 \). We want to set \( 28 = x_0 + dx \), and we want \( dx \) to be relatively small, and we also want \( x_0 \) to be a point for which it is easy to evaluate \( f(x) \). In other words we are looking for a point, \( x_0 \), that is close to 28 and for which the cube root is known. \( x_0 = 27 \) fills the bill.

   So, we have \( f(x) = x^{1/3}, x_0 = 27 \) and \( dx = 28 - 27 = 1 \). Next we compute \( dy \):
   \[
   dy = f'(x_0)dx = \frac{1}{3}x_0^{-2/3}dx = \frac{1}{3}27^{-2/3} \cdot 1 = \frac{1}{27}.
   \]

   Finally, we plug everything back into the approximation formula, above.
   \[
   28^{1/3} \approx 27^{1/3} + dy = 3 + \frac{1}{27} = \frac{82}{27}.
   \]

   **Note:** estimate is within 0.00045 of the true value of \( \sqrt[3]{28} \).

3. Compute the indefinite integrals below.
   a. \[ \int 3x^4 - 2x^3 + 6x^2 + 2x - 1 \, dx = \frac{3}{5}x^5 - \frac{1}{2}x^4 + 2x^3 + x^2 - x + C. \]
   b. \[ \int \sqrt[3]{x} \, dx = \int x^{3/5} \, dx = \frac{5}{8}x^{8/5} + C. \]
   c. \[ \int \frac{3x^2 - 4x + 1}{x^5} \, dx = \int 3x^{-3} - 4x^{-4} + x^{-5} \, dx = -\frac{3}{2}x^{-2} + \frac{4}{3}x^{-3} - \frac{1}{4}x^{-4} + C. \]
4. Find the function \( y = f(x) \), given that \( y' = x - \frac{1}{x} \), and \( f(1) = 3 \).

First, we integrate

\[
\int y' \, dx = \int x - \frac{1}{x} \, dx = \frac{x^2}{2} - \ln |x| + C.
\]

This means that \( f(x) = \frac{x^2}{2} - \ln |x| + C \), and we use the initial value to solve for \( C \).

\[
3 = f(1) = \frac{1^2}{2} - \ln 1 + \frac{1}{2} + C \implies C = 3 - \frac{1}{2} = \frac{5}{2}.
\]

So, \( f(x) = \frac{x^2}{2} - \ln |x| + \frac{5}{2} \).

5. Find the function \( y = g(x) \), given that \( y'' = x^2 - 1 \), \( y'(1) = 2 \) and \( y(1) = 2 \).

First, we solve one initial value problem to find \( y' \), and to do this we begin by integrating \( y'' \).

\[
y' = \int y'' \, dx = \int x^2 - 1 \, dx = \frac{x^3}{3} - x + C_1.
\]

Next, we solve for \( C_1 \) using the initial value for \( y' \),

\[
2 = y'(1) = \frac{1^3}{3} - 1 + C_1 \implies C_1 = 2 + 1 - \frac{1}{3} = \frac{8}{3}.
\]

So, \( y' = \frac{x^3}{3} - x + \frac{8}{3} \), and now we repeat the process to find \( y = g(x) \).

\[
g(x) = \int y' \, dx = \int \frac{x^3}{3} - x + \frac{8}{3} \, dx = \frac{x^4}{12} - \frac{x^2}{2} + \frac{8x}{3} + C_2.
\]

Finally, use the initial data for \( g(x) \) to solve for \( C_2 \),

\[
2 = g(1) = \frac{1}{12} - \frac{1}{2} + \frac{8}{3} + C_2 \implies C_2 = -\frac{1}{4},
\]

giving the final solution \( g(x) = \frac{x^4}{12} - \frac{x^2}{2} + \frac{8x}{3} - \frac{1}{4} \).

6. The demand function \( p = f(q) \) is found by dividing the revenue function, \( r(q) \), by \( q \), i.e., \( p = r/q \). The revenue function is found by solving the initial value problem, \( r' = 200 - q^{2/3} \), \( r(0) = 0 \). First,

\[
r = \int 200 - q^{2/3} \, dq = 200q - \frac{3}{5}q^{5/3} + C.
\]
Next, the initial value \( r(0) = 0 \) implies that \( C = 0 \), so \( r = 200q - \frac{3}{5}q^{5/3} \), and the demand function is

\[
p = 200 - \frac{3}{5}q^{2/3}.
\]

7. Another initial value problem. First,

\[
c = \int (q + 1000)^{1/3} + 50 \, dq = \frac{3}{4} (1000 + q)^{4/3} + 50q + K,
\]

(using the substitution \( u = 1000 + q \) to find the integral). Next, the initial data is \( c(0) = 12000 \) (fixed cost = \( c(0) \)), which we use to solve for the constant of integration, \( K \):

\[
12000 = c(0) = \frac{3}{4} 1000^{4/3} + K = 7500 + K \quad \implies \quad K = 4500.
\]

Thus, the cost function is \( c = \frac{3}{4} (1000 + q)^{4/3} + 50q + 4500 \).