1. (25 points). A box contains 5 red balls, 10 blue balls, and 5 yellow balls. Consider the following experiment. Four balls will be selected at random, without replacement, from the box (the balls will be selected one at a time). The color of each selected ball will be observed, and if the selected ball is red or yellow, 3 new balls of the same color will be put into the box. If the selected ball is blue, no balls will be added to the box.

Obtain the probability that the first two balls will be red, and the third ball will be blue, and the fourth ball will be yellow.

Solution: For $i = 1, ..., 4$, let $R_i$ be the event that a red ball is drawn on the $i$-th draw, $B_i$ be the event that a blue ball is drawn on the $i$-th draw, and $Y_i$ be the event that a yellow ball is drawn on the $i$-th draw. Then, the probability that the first two balls will be red, and the third ball will be blue, and the fourth ball will be yellow can be expressed as $\Pr(R_1 \cap R_2 \cap B_3 \cap Y_4)$.

Using the multiplication rule for conditional probabilities, and the problem assumption regarding red or yellow balls added to the box, we obtain

$$\Pr(R_1 \cap R_2 \cap B_3 \cap Y_4) = \frac{5}{20} \times \frac{7}{22} \times \frac{10}{24} \times \frac{5}{23} = 0.0072.$$ 

2. (25 points). A company uses three different assembly lines, $A_1$, $A_2$, and $A_3$, to manufacture a particular component. Given that a component is manufactured by line $A_1$, it needs rework to remedy a defect with probability 0.05; given that it is manufactured by $A_2$, it needs rework with probability 0.08; and given that it is manufactured by $A_3$, it needs rework with probability 0.1. Suppose that 50 percent of all components are manufactured by line $A_1$; 30 percent are manufactured by line $A_2$; and 20 percent are manufactured by line $A_3$.

A randomly selected component is inspected and it is found that it needs rework. Which one is the most likely line that manufactured it? Which one is the least likely line?

Solution: For $i = 1, 2, 3$, let $A_i$ denote the event that the randomly selected component is manufactured by line $A_i$. Moreover, let $D$ be the event that the component needs rework. It is given that $\Pr(A_1) = 0.5$, $\Pr(A_2) = 0.3$, $\Pr(A_3) = 0.2$, and $\Pr(D \mid A_1) = 0.05$, $\Pr(D \mid A_2) = 0.08$, $\Pr(D \mid A_3) = 0.1$. To find the least and most likely lines, we need to compare $\Pr(A_i \mid D)$, for $i = 1, 2, 3$. Using Bayes theorem, $\Pr(A_i \mid D) = (\Pr(A_i)\Pr(D \mid A_i))/\Pr(D)$. Note that to compare the probabilities $\Pr(A_i \mid D)$, it suffices to compute the numerators $\Pr(A_i)\Pr(D \mid A_i)$, which are given by 0.025 for $i = 1$, 0.024 for $i = 2$, and 0.02 for $i = 3$. Therefore, line $A_1$ is the most likely, and line $A_3$ is the least likely.
3. (25 points). Each one of three contestants on a quiz show is asked to choose one of six possible categories of questions. Assume that \( \Pr(\text{contestant selects category } i) = \frac{1}{6}, \, i = 1, \ldots, 6 \), the same for all three contestants. Suppose that the contestants choose their categories independently of one another. Note that it is possible for two (or all three) contestants to choose the same category. Hence, the sample space consists of outcomes that can be represented as \((x_1, x_2, x_3)\), where \(x_j = 1, \ldots, 6\) is the choice of the \(j\)-th contestant, for \(j = 1, 2, 3\).

(a) (10 points). Are all outcomes of the sample space equally likely? Justify your answer.

Solution: Based on the multiplication rule, the sample space consists of \(6^3\) simple outcomes. Moreover, for a generic simple outcome, we have

\[
\Pr(\{(i, j, k)\}) = \Pr(x_1 = i, x_2 = j, x_3 = k) = \Pr(x_1 = i)\Pr(x_2 = j)\Pr(x_3 = k) = \frac{1}{6^3}
\]

using the assumption of independence. Therefore, all outcomes are equally likely.

(b) (15 points). Obtain the conditional probability that exactly one contestant selects category 1, given that all three contestants select different categories.

Solution: Let \(A\) be the event that exactly one contestant selects category 1, and \(B\) be the event that all three contestants select different categories.

We have

\[
\Pr(B) = \frac{6 \times 5 \times 4}{6^3}
\]

and

\[
\Pr(A \cap B) = \frac{3 \times (1 \times 5 \times 4)}{6^3}
\]

(note that \(\Pr(A \cap B)\) arises through the union of three disjoint events, one for each contestant). Finally, \(\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B) = 0.5\).
4. (25 points). Consider the following game of chance: you win the unit of money you bet if you obtain at least one double-6 (that is, outcome (6,6)) in 24 independent rolls of a pair of balanced dice; if there is no double-6 in the 24 rolls, you lose the unit of money.

(a) (10 points). Your friend (who has not taken AMS 131!) thinks that this is a rather favorable game to the player based on the following argument: the chance of obtaining a double-6 in one roll of the dice is 1/36 and so the probability of winning the game is $24 \times (1/36) = 2/3$. What is the error in this argument?

Solution: Let $B_i$ denote the event that a double-6 is obtained in the $i$-th roll of the dice, for $i = 1, \ldots, 24$. Then, the probability of winning the game can be expressed as the probability of the union $\Pr(B_1 \cup B_2 \cup \ldots \cup B_{24})$ (the union implied by the “at least one double-6” condition). Even though $\Pr(B_i) = 1/36$ for each $i = 1, \ldots, 24$, the error is that these probabilities are summed up to obtain $\Pr(B_1 \cup B_2 \cup \ldots \cup B_{24})$, and this is incorrect because the events $B_i$ are not disjoint.

(b) (15 points). Obtain the correct probability of winning the game.

Solution: This is a setting where working with the complement of the event of interest provides the answer in a much more straightforward manner. In particular, the player loses if there is no double-6 in the 24 rolls, and therefore working with events $B_i$ from part (a), the probability of losing the game is given by

$$\Pr(B_1^c \cap B_2^c \cap \ldots \cap B_{24}^c) = \prod_{i=1}^{24} \Pr(B_i^c) = (35/36)^{24} \approx 0.509$$

using the assumption of independent rolls. Finally, the probability of winning the game is given by $1 - (35/36)^{24} \approx 0.491$. 
