How many ways are there to
pick \(k\) times from a set of \(n\)
objects, where order does not matter,
with replacement?

\[
k = 0 \quad \text{#ways} = 1
\]

\[
k = 1 \quad n.
\]

\[
n = 2.
\]

\[
|\quad|\quad|\quad|\quad|\quad|\quad| \quad k=7
\]

How many ways can I split \(k\)
indistinguishable objects between 2 boxes?

\[
k+1. \quad \text{There could be}
0, 1, 2, \ldots, k \text{ objects}
in the first box.
\]

**Generalize.**

\[
|\quad|\quad|\quad|\quad|\quad|\quad| \quad n=5
\]

\[
|\quad|\quad|\quad|\quad|\quad|\quad| \quad k=8
\]

Equivalent problem:
distribute \(k\) items into \(n\) boxes.
How many ways are there to distribute $k$ items into $n$ boxes?

$$00|0\underbrace{00}_{k \text{ objects}}|0|00$$

$n-1$ separators

How many ways are there to arrange these?

Choose which of the $n+k-1$ positions we put the $k$ items.

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

Choose where to put the $n-1$ separators.

$k = 0$  \hspace{1cm} \binom{n-1}{0} = 1  \hspace{1cm} \binom{n-1}{n-1} = 1$

$k = 1$  \hspace{1cm} \binom{n}{1} = n$

$n = 2$  \hspace{1cm} \binom{k+1}{k} = k+1$.
If choose chair of the after n-1 people left then choose remaining k-1 members from the n-k

\[
\binom{n-k}{k-1} = \frac{(n-k)!}{(k-1)!(n-k-(k-1))!}
\]

prove this by thinking about what those expressions mean.
\[
\binom{M+N}{k} = \sum_{j=0}^{k} \binom{M}{j} \times \binom{n}{k-j}
\]

choose \( j \) items of type 1

\[M\] ways

\[\binom{M}{j}\] ways

\[\binom{n}{k-j}\] ways.

add up the options for \( j = 0, 1, \ldots, k \).

---

\text{Birthday Problem.}

How large must a group of \( n \) people be for (at least) two of them to share a birthday \( \mathcal{B} \) with prob. 0.5

or:

In a group of \( n \) people, what's the probability that no two of them share a birthday?

\( n = 180 \)
\( n = 30 \)
\[ P(\text{no two people share a birthday}) = 0 \quad n \geq 366 \]

\[
\begin{array}{c}
\frac{n \leq 365}{365 \times 364 \times \frac{363}{2} \times \ldots \times (365-n+1)}
\end{array}
\]

\[ P(\text{match}) = 1 - P(\text{no match}) \]

\[
\begin{array}{c}
\leq 50.7\% & n = 23 \\
97\% & n = 50 \\
> 99.99\% & n = 100.
\end{array}
\]

\[
\binom{n}{2} = \frac{n(n-1)}{2} = \frac{23 \times 22}{2} = 253
\]

\# of pairs / \# potential matches.
Non-rigorous Definition of Probability

- not all outcomes are equally likely
- not finitely many outcomes

A Probability Space

\[ \begin{align*}
S & \text{ - sample space} \\
P & \text{ - function which takes an event } A \subseteq S, \text{ and returns } P(A) \in [0, 1] \\
P(\emptyset) &= 0 \\
P(S) &= 1
\end{align*} \]

\[ P \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P(A_n) \]

if \( A_n, n=1, 2, \ldots \) are disjoint.
Proposition:

\[ P(A^c) = 1 - P(A) \]

\[
1 = P(S) = P(A \cup A^c)
\]

\[ = P(A) + P(A^c) \quad \text{as } A, A^c \text{ are disjoint.} \]

\[ P(A^c) = 1 - P(A) \]

2. If \( A \subseteq B \)

\[ P(A) \leq P(B) \]

\[
B = A \cup (B \cap A^c)
\]

the "ring" of \( B \) that's not in \( A \)

\[ \text{disjoint.} \]

\[ P(B) \leq P(A) \]

\[
P(B) = P(A) + P(B \cap A^c).
\]

\[ \geq P(A) \]
\[ P(A \cup B) \text{ when } A, B \text{ not disjoint} \]

\[ = P(A) + P(B) - P(A \cap B) \]

\[ P(A \cup B) = P(A \cup (B \cap A^c)) \]

\[ \underbrace{\text{part of } B \text{ that's not in } A} \]

\[ = P(A) + P(B \cap A^c). \]

Assuming the result to be true, we require

\[ P(B) - P(A \cap B) = P(B \cap A^c). \]

\[ P(B) = P(B \cap A) + P(B \cap A^c). \text{ True.} \]

Hence,

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B). \]

3 events,

\[ P(A \cup B \cup C) \]

\[ = P(A) + P(B) + P(C) \]

\[ - P(A \cap B) - P(A \cap C) - P(B \cap C) \]

\[ + P(A \cap B \cap C) \]
\[ P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) \]
\[ = \sum_{j=1}^{n} p(A_j) \]
\[ - \sum_{i<j} p(A_i \cap A_j) \]
\[ + \sum_{i<j<k} p(A_i \cap A_j \cap A_k) \]
\[ - \ldots \]
\[ + (-1)^{n+1} p(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) \]

For \( n = 2 \) (above), subtract \( p(A_1 \cap A_2) \)
For \( n = 3 \), add \( p(A_1 \cap A_2 \cap A_3) \)
Example.

n letters, n envelopes

put each letter in a randomly chosen envelope.

what's the probability that at least one letter ends up in the correct envelope?

let $A_j$ be the event that the letter for person $j$ ends up in the $j^{th}$ envelope.

$$P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) = \sum_{j=1}^{n} P(A_j) \geq n \times \frac{1}{n}$$

$$- \sum_{i<j} P(A_i \cap A_j) \binom{n}{2} \frac{1}{n} \times \frac{1}{n-1}$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \binom{n}{3} \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$$

$$\vdots$$

$$\vdots$$

$$N = 10 \quad p( ) = 0.2$$

$$N = 10,000 \quad 0.05$$

$$= 0.98$$

$$= 0$$
\begin{align*}
\binom{n}{2} \frac{1}{n(n-1)} &= \frac{n!}{(n-2)! \cdot 2!} \times \frac{1}{n(n-1)} = \frac{1}{2}, \\
\binom{n}{3} \frac{1}{n(n-1)(n-2)} &= \frac{n!}{(n-3)! \cdot 3!} \times \frac{1}{n(n-1)(n-2)} = \frac{1}{3!}.
\end{align*}

\[ P(A_1 \neq A_2 \neq A_3 \ldots \neq A_n) = 1 - \frac{1}{2^1} + \frac{1}{3^1} - \frac{1}{4^1} + \ldots + (-1)^{n-1} \frac{1}{n^1} \]

\[ X \sim 1 - \frac{1}{e} \]

\[ P(\text{no one gets the right letter}) = \frac{1}{e} = 0.37 \]

- non intuitive, especially as \( n \to \infty \)
Independence.

2 events - one event gives you no information about the other.

Defn: Events $A$ and $B$ are independent

if $\quad P(A \cap B) = P(A) \times P(B)$.

Note: This is not the same as disjoint.

If $A$ occurs

$B$ cannot occur.

$A, B, C$ are independent if

$p(A, B) = p(A)p(B)$
$p(A, C) = p(A)p(C)$
$p(B, C) = p(B)p(C)$.  

Pairwise independent

$p(A, B, C) = p(A)p(B)p(C)$

$A_1, \ldots, A_n$. Any 2 must be independent

$\begin{align*}
3 \\
4 \\
\vdots \\
n
\end{align*}$
Fair dice, which is most likely?

A) At least 1 six on 6 rolls
B) At least 2 sixes on 12 rolls
C) At least 3 sixes on 18 rolls.

\[ P(\text{at least 1 6 on 6 rolls}) = 1 - P(\text{no sixes on 6 rolls}) \]
\[ = 1 - \left( \frac{5}{6} \right)^6 \]
\[ = 0.665 \]

\[ P(B) = 1 - P(\text{no sixes on 12 rolls}) - P(\text{one six on 12 rolls}) \]
\[ = 1 - \left( \frac{5}{6} \right)^{12} - 12 \times \frac{1}{6} \times \left( \frac{5}{6} \right)^{11} \]
\[ = 0.619 \]

\[ P(C) = 1 - \sum_{k=0}^{2} P(\text{exactly k sixes on 18 rolls}) \]
\[ = 1 - \sum_{k=0}^{2} \binom{18}{k} \left( \frac{1}{6} \right)^k \left( \frac{5}{6} \right)^{18-k} \]
\[ = 0.597 \]

\[ P(A) > P(B) > P(C) \]