Conditional Probability.

\[ P(A | K_1) \]
\[ P(A | K_1, K_2) \]

**Definition.**

\[ P(A | B) = \frac{P(A \cap B)}{P(B)} \quad \text{for } P(B) > 0 \]

Probability of \( A \) given \( B \)

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**Frequency definition.**

- \( B \rightarrow 1 \ 1 \ 1 \ 2 \ 4 \ 2 \ 6 \ \Rightarrow A \)
- \( B \rightarrow 2 \ 3 \ 4 \ 2 \ 3 \ 4 \ 6 \)
- \( 4 \ 5 \ 5 \ 4 \ 3 \ 4 \ 2 \)
- \( 1 \ 5 \ 3 \ 4 \ 2 \ 4 \ 1 \)

B - one of the rolls is a 6
A - one of the rolls is a 1

\[ P(A | B) = \frac{\text{times both } A \text{ and } B \text{ occur}}{\text{times } B \text{ occurs}} \]
\[ P(A \cap B) = P(B) P(A | B) \]
\[ = P(A) P(B | A) \]

\[ P(A_1 \ldots A_n) = P(A_1) P(A_2 | A_1) P(A_3 | A_1, A_2) \ldots P(A_n | A_1 \ldots A_{n-1}) \]
\[ = P(A_n | A_1 \ldots A_{n-1}) \ldots P(A_2 | A_1, A_2) P(A_1 | A_1) P(A_1) \]

\[ P(A_1 \cap A_2) = \text{write or } P(A_1, A_2) \]

\[ P(B | A) = \frac{P(A | B) P(B)}{P(A)} \]
Bayes' Theorem.

\[ \uparrow \text{Inductive Inference.} \]
\[ P(B) = ? \]

Partition \( S \) into \( A_1, \ldots, A_n \).

\[ P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n) \]

All \( B \cap A_i \) are disjoint.

\[ P(B) = P(B | A_1)P(A_1) + P(B | A_2)P(A_2) + \ldots + P(B | A_n)P(A_n) \]

Law of Total Probability.

How useful this is depends on the choice of the partitions \( A_1, \ldots, A_n \).

Testing for a disease.

Disease affects \( n \) 1\% of the population

Patient tests positive.

D: This specific patient has the disease.

T: This specific patient tests positive.
\[ P(D | T) \] — the patient cares about the prob. that they have the disease, given a positive test.

This is not the same as \( P(T | D) \).

\[ P(T | D) = \text{this is the accuracy of the test} \]
\[ = 0.95 \text{ (say)} \]
\[ P(T^c | D^c) = 0.95 \text{ (say)} \]

\[ P(D | T) = \frac{P(T | D) P(D)}{P(T)} \]

\[ = \frac{P(T | D) P(D)}{P(T | D) P(D) + P(T | D^c) P(D^c)} \]

\[ \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99} = 0.16. \]
95% accurate test
but patient only has 16% chance of having the disease.

Intuition - interplay between how rarely the test is wrong and how rarely someone has the disease.

Frequency definition:

1000 patients 10 have the disease.
990 don't have the disease.
but 5% of them will test positive
> 50 people.

\[
p(\text{someone who tests positive does have the disease}) = \frac{\# \text{positive with disease}}{\# \text{positive}}
\]

\[
= \frac{10}{10 + 50} = \frac{1}{6} \approx 16%
\]
Examples.

Random 2 card hand from standard 52 card deck.

\[ P(\text{both cards are aces} \mid \text{have ace}) \]

\[ P(\text{both cards are aces} \mid \text{have ace of spades}) \]

\[ P(\text{both aces} \mid \text{have ace}) = \frac{P(\text{both aces and have ace})}{P(\text{have ace})} \]

\[ = \frac{P(\text{both aces})}{P(\text{have ace})}. \]

\[ = \frac{\binom{4}{2}/52}{\binom{52}{2}} \cdot \frac{1 - P(\text{neither card is an ace})}{1 - (\binom{4}{2}/52)} \]

\[ = \frac{\binom{4}{2}/52}{\frac{1}{32}} = \frac{1}{32} \]

\[ P(\text{both aces} \mid \text{have ace of spades}) \]

\[ = \frac{3}{51} = \frac{1}{17} \]

It's twice as likely that both are aces if you know you have the ace of spades than if you just know that you have an ace.
Common Conditional Probability Errors.

1/ Confusing \( P(A|B) \) with \( P(B|A) \) - "prosecutor's fallacy".

- \( A \) - person is pregnant
- \( B \) - person is a woman.

\[
P(A|B) = P( \text{person is pregnant} | \text{woman} ) = 0.02 - 0.03
\]

\[
P(B|A) = P( \text{person is woman} | \text{pregnant} ) = 1.
\]

2/ Confusing \( P(A) \) "prior"

with \( P(A|B) \) "posterior"

"given that A occurs" does not imply \( P(A) = 1 \)

does imply \( P(A|A) = 1 \)

3/ Confusing independence and conditional independence.

Definition: Events \( A \) and \( B \) are conditionally independent given \( C \) if

\[
P(A \cap B | C) = P(A|C)P(B|C)
\]
Does independence imply conditional independence? - No

Does conditional independence imply independence? - No.

Example: playing cards against an opponent of unknown strength.

conditional on the strength of the opponent, as outcomes of the series of games are independent.

however the games are not independent - early games give you information about the outcomes of later games (unconditional independence means that earlier games tell you nothing about later games).
Does independence imply conditional independence?

- phenomena with multiple causes.

A - fire alarm goes off.

2 possible causes:
- F - there is a fire
- T - someone has burned their toast.

\[ P(F | A, T^c) = 1 \]

F and T are not conditionally independent given A.
Monty Hall Problem.

3 doors (identical)
1 door has a car.
2 doors have goats.

Assume you want the car.
Assume Monty knows which door has the car and which have goats.

Chose a door.
Monty opens one of the other doors to reveal a goat.
Monty offers for you to switch your choice to the other closed door.

Should you switch?

→ Monty always opens a goat door.
  if he has a choice, opens either door with equal probability.
Tree:

\[
\begin{align*}
1 & \quad 2 \quad 3 \\
\frac{1}{3} & \quad \frac{1}{3} \quad \frac{1}{3}
\end{align*}
\]

Condition on Monty opening door 2, consider paths indicated.

- The bottom path (car is behind door 2) is twice as likely as the top path (car is behind door 1).

\[
\begin{align*}
\frac{2}{3} & \quad \frac{1}{3} \\
\frac{1}{3} & \quad \frac{1}{3}
\end{align*}
\]

\[\Rightarrow \text{Switch.}\]

Car is equally likely to be behind each of the doors.

\[\Rightarrow \frac{2}{3}\] of the time, Monty Hall has no choice of which door to open, and by opening a door, he tells you where the car is.

\[\frac{1}{3}\] of the time, you were right to begin with. \[\Rightarrow \text{Switch.}\]
Assume car is behind door 1.

\[ P(s|D_1) = 0 \]
\[ P(s|D_2) = 1 \]
\[ P(s|D_3) = 1 \]

\[ P(s) = \frac{2}{3} \Rightarrow \text{switching strategy gives } \frac{2}{3} \text{ chance of getting the car.} \]

Homework: Use Bayes' Theorem to solve the Monty Hall Problem.
Simpson's Paradox

- please read.

Section 10.5