AMS131-02 - Quiz 2
Tuesday 20th May, 2014.
You must show working/explain all answers for full credit.

1. Suppose that the proportion \( \theta \) of defective items in a large manufactured lot is unknown, and the prior pdf for \( \theta \) is given by

\[
f(\theta) = \begin{cases} 
2(1-\theta) & \text{for } 0 < \theta < 1 \\
0 & \text{otherwise}
\end{cases}
\]

When 8 items are selected at random from the lot, it is found that exactly three of them are defective. Determine the posterior distribution for \( \theta \).

\[
f(\theta | d) = \frac{f(d | \theta) f(\theta)}{f(d)}
\]

\[
P(d | \theta) \sim \text{Binomial}(8, \theta)
\]

\[
p(d | \theta) = \binom{8}{3} \theta^3 (1-\theta)^5
\]

\[
f(\theta) \text{ is given as } 2(1-\theta) \text{ for } 0 < \theta < 1
\]

Here

\[
f(d | \theta) \propto \theta^3 (1-\theta)^5 \times (1-\theta)
\]

\[
= \theta^4 (1-\theta)^6
\]

This has the form of a Beta distribution

\[
\text{with } a - 1 = 3 \quad b - 1 = 6 \quad \Rightarrow \quad a = 4 \quad b = 7.
\]

Here

\[
f(\theta | d) = \frac{n(d) \theta^b (1-\theta)^a}{\Gamma(n) \Gamma(\pi)}
\]

\[
\Gamma(n) = (n-1)!
\]

\[
f(d | \theta) = \binom{8}{3} \theta^3 (1-\theta)^5
\]

\[
= \frac{10 \times 9 \times 8 \times 7 \times \theta^3 (1-\theta)^5}{3 \times 2}
\]

\[
= \frac{5 \times 3 \times 8 \times 7 \times \theta^3 (1-\theta)^5}{3 \times 6}
\]

\[
= \frac{5 \times 8 \times 3 \times 6 \times \theta^3 (1-\theta)^5}{3 \times 6}
\]

\[
= 8 \times 6 \times \theta^3 (1-\theta)^5
\]

\[
= 8 \times 6 \theta^3 (1-\theta)^5
\]
2. Suppose that $X$ and $Y$ are independent random variables with the following p.d.f.s:

$$f_1(x) = \begin{cases} 
1 & \text{for } 0 < x < 1, \\
0 & \text{otherwise},
\end{cases}$$

$$f_2(y) = \begin{cases} 
8y & \text{for } 0 < y < \frac{1}{2}, \\
0 & \text{otherwise}.
\end{cases}$$

Determine the value of $P(X > Y)$.

\[ P(X > Y) = \iint_{\text{shaded region}} 8y \, dx \, dy. \]

\[ = \int_{y=0}^{1/2} \int_{x=y}^{1} 8y \, dx \, dy \]

\[ = \int_{y=0}^{1/2} 8y \left[ x \right]_{x=y}^{1} \, dy \]

\[ = \int_{y=0}^{1/2} 8y \left( 1 - y \right) \, dy \]

\[ = \int_{y=0}^{1/2} 8y - 8y^2 \, dy \]

\[ = \left[ 4y^2 - \frac{8y^3}{3} \right]_{y=0}^{1/2} \]

\[ = 4 \left( \frac{1}{4} - \frac{1}{24} \right) \]

\[ = 1 - \frac{1}{3} \]

\[ = \frac{2}{3}. \]