1. \[ E[(x+y)^2] = E[x^2 + 2xy + y^2] \]
   \[ = E[x^2] + 2E[xy] + E[y^2]. \]

   \( x, y \) are iid so \[ E[y^2] = E[x^2] \]
   and \[ E[xy] = E[x]E[y] \]
   and \[ E[y] = E[x] \]

   \[ = 2E[x^2] + 2(E[E])^2 \]

2. \[ X = X_1 + X_2 + \ldots + X_r \]

   \( X_1 \): \# failures before 1st success.

   \( X_2 \): \# failures before 2nd success.

   \( \vdots \)

   \( X_r \): \# failures before \( r \)th success.

   The \( X_i \)'s are iid Geometric (p)

   \[ \text{Var}(X) = \text{Var}(X_1 + X_2 + \ldots + X_r) \]

   \[ = \text{Var}(X_1) + \text{Var}(X_2) + \ldots + \text{Var}(X_r) \]

   Variance of a Geometric (p) is \[ \frac{1-p}{p^2} \]

   \[ \Rightarrow \text{Variance of Negative Binomial (r, p)} = \frac{r(1-p)}{p^2} \]
3. \( X, Y \) iid \( N(0, 1) \).

\[ w = x^2 + y^2 \]

\[ f_w(w) = \frac{1}{2} e^{-w/2} \]

a) \( R^2 = w \)

\[ R = w^{1/2} \]

\[ f_r(r) dr = f_w(w) dw \]

\[ f_r(r) = f_w(w) \frac{dw}{dr} \]

\[ \frac{dw}{dr} = 2R = 2r \]

\[ = \frac{1}{2} e^{-r^2/2} \cdot 2r = r e^{-r^2/2} \]

b) \( P(x > 2y + 3) \)

\[ = \Phi(x - 2y > 3) \]

RV \( x - 2y \) is Normal, mean \( \mu = 0 \)

\[ \text{variance} = 5 \]

Hence \( P(x - 2y > 3) \) is same as \( N(0, 5) > 3 \)

\[ = 1 - \Phi \left( \frac{3}{\sqrt{5}} \right) \]
4.

a) \[ E(U_1) = \frac{1}{2} \]
\[ \text{Var}(U_1) = \frac{1}{12} \]

Mean of \( X \): \[ E(X) = E(U_1) + E(U_2) + \ldots + E(U_{60}) \] (linearity of expectations)
\[ = 60 \, E(U_1) \]
\[ = 30 \]

\[ \text{Var}(X) = \text{Var}(U_1) + \text{Var}(U_2) + \ldots + \text{Var}(U_{60}) \] (as \( U_i \)'s are independent)
\[ = \frac{60}{12} = 5 \]

b) \[ P(X > 17) \]

Approximate pdf of \( X \) by \( \mathcal{N}(30, 5) \).

\[ P(X > 17) = P(Z > \frac{17 - 30}{5}) \]
\[ = P(Z > \frac{-13}{\sqrt{5}}) \]
\[ = P(Z > -2.8) \]
\[ = 0.9975 \] (area to the left is approx. 1)
Number the 48 non-aces 1 through 48.

Let $X_i$ be indicator that card with number $i$ is dealt before any of the aces.

$$X = X_1 + X_2 + \ldots + X_{48}$$

is total # of cards before first ace.

and, by using linearity of expectation and symmetry,

$$E[X] = 48 \cdot E[X_1].$$

**Expectation of an indicator RV is the prob. of the indicator RV taking the value 1.**

What's the prob of card 1 being dealt before any of the 4 aces?

Consider the orderings of the 5 cards. One of them results in card 1 being dealt before any of the 4 aces.

$$\Rightarrow E[X_1] = \frac{1}{5}$$

$$\Rightarrow E[X] = \frac{48}{5}$$
\[ p(\text{1st six appears on roll } k) = \left( \frac{5}{6} \right)^{k-1} \left( \frac{1}{6} \right) \quad k = 1, 2, \ldots \]

Event that Bob wins in all odd \( k \).

These are disjoint, so

\[ p(\text{Bob wins}) = \sum_{k \text{ odd}} p(\text{1st six appears on roll } k) \quad \text{odd} \]

\[ = \sum_{j=1}^{\infty} \left( \frac{5}{6} \right)^{2j-2} \left( \frac{1}{6} \right) \]

\[ = \frac{1}{6} \sum_{j=1}^{\infty} \left( \frac{25}{36} \right)^{j-1} \]

\[ = \frac{1}{6} \sum_{j=0}^{\infty} \left( \frac{25}{36} \right)^{j} \]

\[ = \frac{1}{6} \cdot \frac{1}{1 - \frac{25}{36}} \]

\[ = \frac{6}{11} \]
7. \[ \binom{5}{3} \left( \frac{1}{3} \right)^3 \left( \frac{2}{3} \right)^2 \]
\[= \binom{10}{6}.\]

8. a) \[P(\text{dry on at least one of next 3 days})\]
\[= 1 - P(\text{rain on all 3 of next 3 days}).\]
\[= 1 - P(\text{wet tomorrow} \mid \text{wet today}) \times P(\text{wet in 2 days hence} \mid \text{wet tomorrow}) \]
\[\times P(\text{wet in 3 days hence} \mid \text{wet tomorrow})\]
\[= 1 - 0.4 \times 0.4 \times 0.4\]
\[= 0.936.\]

b) For Friday, it is given by
\[S^2 = T^2\]
\[S^T = \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.62 & 0.38 \\ 0.62 & 0.38 \end{bmatrix}\]
\[S^T 2 = \begin{bmatrix} 0.62 & 0.38 \\ 0.62 & 0.38 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.662 & 0.338 \\ 0.662 & 0.338 \end{bmatrix}\]
\[\Rightarrow \text{Prob. (wet on Friday)} = 0.338.\]