Statistical Inference
Classical and Bayesian Methods

Class 11

AMS-UCSC

Tu Feb 21, 2012
Topics

We will talk about...

1. Exchangeability
Many statistical applications involve multiple parameters that are connected following a certain structure.

The joint probability model for these parameters should reflect this connection.

Generally observable outcomes are modeled conditionally on certain parameters.

The parameters are given a probabilistic specification in terms of further parameters called the hyperparameters.

This hierarchical thinking helps to understand the multiparameter problem and plays an important role in computational strategies.
Study of the effectiveness of a cardiac treatment

- Assume that patients of hospital $j$ have a survival probability $\theta_j$ of a given cardiac treatment.
- It is expected that estimates of $\theta_j$ would represent a sample of hospitals and they should be related to each other.
- This could be achieved in a natural way if we assume a prior distribution in which the $\theta_j$ are viewed as samples from a common population.
- We have data $y_{ij}$ where $i$ indicates the patient and $j$ indicates the hospital. The values of $\theta_j$ are not themselves observed, but data $y_{ij}$ is used to estimate aspects of the population distribution of $\theta_j$. 
Consider a set of experiments $j = 1, \ldots, J$, in which experiment $j$ has a data $y_j$ (a vector) and a parameter vector $\theta_j$ with likelihood $f(y_j|\theta_j)$ (the same concept can be applied to non-experimental data).

**Example:** Suppose each vector $y_j$ is a sample from a normal distribution with mean $\mu_j$ and common variance $\sigma^2$. In this case $\theta_j = (\mu_j, \sigma^2)$.

The goal is to create a joint probability distribution for all parameters $\theta$. In order to do this we use the exchangeability concept.

**Definition**

The parameters $\theta = (\theta_1, \theta_2, \ldots, \theta_J)$ are exchangeable in their joint distribution if $\xi(\theta_1, \ldots, \theta_J)$ is invariant to permutations of the indexes $(1, \ldots, J)$.

In general, the less we know about a problem, the more confident we can be about the assumption of exchangeability.
**Definition**

**Examples**

Simplest form of an exchangeable distribution

We consider each of the parameters $\theta_j$ as an independent sample from a prior distribution governed by some unknown vector $\phi$. We can write:

$$\xi(\theta|\phi) = \prod_{j=1}^{J} \xi(\theta_j|\phi)$$

Since in general $\phi$ is unknown, our distribution for $\theta$ must take into account the uncertainty in $\phi$.

We need to integrate out with respect to $\phi$:

$$\xi(\theta) = \int \xi(\theta, \phi) d\phi = \int \left[ \prod_{j=1}^{J} \xi(\theta_j|\phi) \right] \xi(\phi) d\phi$$

This integral is a mixture of iid distributions and it is normally used to capture exchangeability in practice.
Full Bayesian description

The hyperprior distribution

Since the parameter vector $\phi$ is unknown, it has its own prior distribution. We should look for the posterior distribution of the vector $(\phi, \theta)$.

- The joint prior distribution can be written as:

$$\xi(\phi, \theta) = \xi(\phi) \cdot \xi(\theta|\phi)$$

- The joint posterior distribution can be written as:

$$\xi(\phi, \theta|y) \propto \xi(\phi, \theta) f(y|\phi, \theta) = \xi(\phi, \theta) f(y|\theta)$$

This last simplification is because $f(y|\phi, \theta)$ depends only on $\theta$. Its dependence on $\phi$ is through its dependence on $\theta$.

Note: In previous examples we assumed $\phi$ was known. Now we assume it is unknown.
• Since we need the joint prior distribution for \((\phi, \theta)\), we need to set a prior distribution for \(\phi\). This is the hyperprior distribution.

• In most practical problems we do not have enough information to set an informative prior distribution for \(\phi\).

• In that case we can set a simple non-informative prior distribution on \(\phi\)

• However we should try to find more prior information if we find too much variation in the posterior distribution.
Thanks for your attention ...