Statistical Inference
Classical and Bayesian Methods

Class 8

AMS-UCSC

Th Feb 2, 2012
1. Equivalence of Tests and Confidence intervals
Equivalence of Tests and Confidence Intervals

Example: A confidence interval for the Mean of a Normal distribution

Confidence intervals can be used as alternative methods to report the results of a test of hypotheses.

Suppose we have a normal sample from a normal distribution with unknown mean $\mu$ and unknown variance $\sigma^2$.

Previously we used the statistics

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad \sigma' = \left( \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2}{n - 1} \right)^{1/2}$$

The coefficient $\gamma$ confidence interval for $\mu$ is the interval:

$$\left( \bar{X}_n - T_{n-1}^{-1}((\gamma + 1)/2)\sigma' / n^{1/2}, \bar{X}_n + T_{n-1}^{-1}((\gamma + 1)/2)\sigma' / n^{1/2} \right)$$

where $T_{n-1}^{-1}(.)$ is the quantile function of the $t$ distribution with $n - 1$ degrees of freedom.
Equivalence of tests and Confidence sets

Example: A two-sided confidence interval for the Mean of a Normal distribution
(Cont.)

The t-test:
We can use the previous interval to find a level $\alpha_0 = 1 - \gamma$ test of the hypotheses:

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

The test will reject the hypothesis $H_0$ if $\mu_0$ is not in the interval

$$(\bar{X}_n - T_{n-1}^{-1}((\gamma + 1)/2)\sigma'/n^{1/2}, \bar{X}_n + T_{n-1}^{-1}((\gamma + 1)/2)\sigma'/n^{1/2})$$

$\mu_0$ is not in the interval if and only if

$$|n^{1/2}(\bar{X}_n - \mu_0)/\sigma'| \geq T^{-1}(\frac{1+\gamma}{2})$$
Equivalence of tests and Confidence sets

Example: Case of the one-sided confidence interval for the Mean of a Normal distribution

The random interval \((A, \infty)\) defined by the \(A = \bar{X}_n - T_{n-1}(\gamma)\sigma'/\sqrt{n}\) statistic such that

\[ P(A < \mu) \geq \gamma \]

is a one-sided coefficient \(\gamma\) confidence interval for \(\mu\), or a one-sided 100\(\gamma\) percent confidence interval for \(\mu\). It is possible to test that \(\mu_0\) is not in this interval if and only if \(\bar{X}_n \geq \mu_0 + \sigma'\sqrt{n^{-1}/2} T_{n-1}(\gamma)\). We can use this interval to test the hypothesis:

\[ H_0 : \mu \leq \mu_0, \quad H_1 : \mu > \mu_0 \]
Example: Rain from seeded clouds
Case of the one-sided confidence interval for the Mean of a Normal distribution

Example: Rainfall from seeded clouds
Suppose the average of 26 log-rainfalls is $\bar{X}_n = 5.134$ and the observed value of $\sigma' = 1.6$
We want to find the 90% lower confidence for $\mu$. We get
$$T_{25^{-1}}(0.9) = 1.316.$$  
The observed lower confidence limit is:
$$a = 5.134 - 1.316 \frac{1.6}{26^{1/2}} = 4.727$$
Since $\mu_0 = 4$ is not in the observed confidence interval, we would reject $H_0: \mu \leq 4$ at a level $\alpha_0 = 0.1$
Example

Length of Fibers

We have measurements of the length in millimeters of metal fibers produced by a certain process. They are assumed normally distributed with unknown mean \( \mu \) and unknown variance \( \sigma^2 \). The following hypotheses are to be tested:

\[
H_0 : \quad \mu \leq 5.2 \\
H_1 : \quad \mu > 5.2
\]
Equivalence of Tests and Confidence intervals

Example
Length of Fibers (Cont.)

We should carry a t-test with level of significance $\alpha_0$. If $n = 15$, $\bar{X}_{15} = 5.4$, $\alpha_0 = 0.05$ and $\sigma' = 0.4226$, the statistic

$$U = n^{1/2}(\bar{X}_n - \mu_0)/\sigma'$$

has a $t$ distribution with 14 degrees of freedom when $\mu_0 = 5.2$

From the $t$-distribution table $T_{14}^{-1}(0.95) = 1.761$. The null hypothesis will be rejected if $U > 1.761$. The numerical value of $U = 1.833$. This implies that $H_0$ is rejected at a 0.05 (5%) confidence level.
Example

Length of Fibers (Cont.)

p-value

You can calculate the tail area to the right of the value $U = 1.833$ using a computer software (e.g. R).

In R:

```r
> 1 - pt(1.833, 14)
[1] 0.04407537
```

The $p$-value is 0.0441.
Thanks for your attention ...