AMS 132: Classical and Bayesian Inference
Exam 2 (Thursday February 26, 2015)

This is a closed-book, closed-notes exam, with the exception of one (letter size) piece of paper with formulas on both sides. You can use without proof any result developed in class or included in the textbook sections that correspond to the material for this exam. Any other result you use for a problem solution must be derived. Please show all your work and indicate your answers in the spaces provided. If you need more space, use the back of the sheet on which the problem appears. Please note that unsupported answers will receive little (or no) credit, even if they are correct. Good luck!

1. (20 points). Assume that $X_1, \ldots, X_n$ is a random sample from a normal distribution with unknown mean $\mu$ and known variance $\sigma^2 = 1$. How large must the sample size $n$ be in order for the confidence interval for $\mu$, with confidence coefficient 0.9, to have length less than 0.1? (It is given that $\Phi(1.645) = 0.95$, that is, the value of the standard normal distribution function at 1.645 is 0.95.)
2. (20 points). Suppose that, conditional on (unknown) mean parameter $\theta > 0$, $X_1, ..., X_n$ form a random sample from a Poisson distribution with probability function

$$f(x \mid \theta) = \frac{e^{-\theta} \theta^x}{x!}, \quad x = 0, 1, 2, ...$$

(a) (10 points). What is the posterior distribution for $\theta$ under an exponential prior distribution with mean 1?

(b) (10 points). Consider a specific data set, with sample size $n = 10$ and sample mean 201.8, assumed to arise as a random sample from a Poisson distribution with mean $\theta$. Compare for this data set the M.L.E. $\hat{\theta} = \bar{x}$ with the posterior expectation for $\theta$, the latter obtained under the prior from part (a). What is the reason (or reasons) for the difference between the values of the M.L.E. and the posterior expectation?
3. (30 points). Suppose that $X_1, \ldots, X_n$ form a random sample from the Rayleigh distribution, which is a continuous distribution with probability density function

$$f(x \mid \theta) = \theta x \exp(-0.5\theta x^2), \quad \text{for } x > 0$$

where $\theta > 0$ is the (unknown) parameter of the distribution.

(a) (10 points). Obtain the Fisher information $I_n(\theta)$ in the random sample $(X_1, \ldots, X_n)$.

(b) (20 points). Derive the M.L.E. of $\theta$. Obtain the asymptotic confidence interval, with confidence coefficient $\gamma$, for $\theta$ based on the large-sample normal approximation to the distribution of the maximum likelihood estimator of $\theta$. 
4. (30 points). Consider Bayesian inference for the same distribution with problem 3, that is, now conditional on \( \theta > 0 \), the \( X_1, ..., X_n \) form a random sample from the Rayleigh distribution with probability density function \( f(x \mid \theta) = \theta x \exp(-0.5x^2) \), for \( x > 0 \). Consider a gamma distribution with parameters \( \alpha \) and \( \beta \) as the prior distribution for \( \theta \), that is, \( \xi(\theta) \propto \theta^{\alpha-1} \exp(-\beta \theta) \), for \( \theta > 0 \).

(a) (10 points). Show that the gamma distribution above provides the conjugate prior for \( \theta \), and obtain the parameters of the posterior distribution given data \( x = (x_1, ..., x_n) \).

(b) (10 points). What is \( \text{E}(\theta \mid x) \), that is, the posterior expectation for \( \theta \)? When will the M.L.E. of \( \theta \) be close to \( \text{E}(\theta \mid x) \)?

(c) (10 points). Derive the expression for the posterior predictive density, \( f(x_{n+1} \mid x) \), corresponding to future response \( X_{n+1} \) with (unobserved) value \( x_{n+1} \).