1. Introduction

In this Assignment, you will use Matlab to find numerical approximations to systems of first order initial value problems of the form

\[ \begin{align*}
    y_1' &= f_1(t, y_1, y_2, \ldots, y_k) \\
    y_2' &= f_2(t, y_1, y_2, \ldots, y_k) \\
    &\vdots \\
    y_k' &= f_k(t, y_1, y_2, \ldots, y_k)
\end{align*} \]

with initial conditions

\[ y_1(t_0) = y_1^*, \quad y_2(t_0) = y_2^*, \ldots, \quad y_k(t_0) = y_k^*. \]

In some cases, these systems appear naturally as models for various phenomena. In other cases, they arise from the study of equations, or systems of equations of higher order.

2. The Assignment

(*) Read the third MATLAB note on the Supplements page, to learn about using ode45 with vector-valued initial value problems and plotting phase portraits and direction fields for such systems.

1. One of the simplest models for the interaction of predator and prey species is given by the Lotka-Volterra equations:

\[ \begin{align*}
    \frac{dx}{dt} &= r_1 x - \alpha xy \\
    \frac{dy}{dt} &= \beta xy - r_2 y
\end{align*} \]

where, in this nonlinear model

- \( x(t) \) is the size of the prey population at time \( t \) and \( y(t) \) is the size of the predator population at time \( t \).
- The parameters \( r_1, r_2, \alpha \) and \( \beta \) are all assumed to be positive, as explained below.
- In the absence of predators \( (y = 0) \), the prey population is assumed to grow exponentially, hence the term \( r_1 x \) on the right hand side of the first equation.
- In the absence of prey \( (x = 0) \), the predator population is assumed to decline exponentially (via death or migration), hence the \( -r_2 y \) term on the right hand side of the second equation.
- If both populations are positive, then \( xy \) is a measure of the frequency with which the two populations interact. These interactions have a negative effect on the growth rate of the prey population, hence the \( -\alpha xy \) term on the right hand side of the first equation. Likewise, these interactions have a positive effect on the growth rate of the predator population, hence the \( \beta xy \) term on the right hand side of the second equation.

(a) Plot a direction field for the Lotka-Volterra system with \( r_1 = 0.4, \ r_2 = 0.3, \ \alpha = 0.1 \) and \( \beta = 0.05 \). You should plot the direction field in the first quadrant, with \( 0 < x < 10 \) and \( 0 < y < 10 \).

(b) Use ode45 to compute the solution to the same system in the interval \([0, 40]\), assuming that \( x(0) = 10 \) and \( y(0) = 5 \).

(c) Plot the graphs of the solutions \( x(t) \) and \( y(t) \) in the same figure (using different colors), and describe how the populations evolve over time. What are the periods and amplitudes of populations?
(d) Repeat (b) and (c) with the same values of \( r_1, r_2, \alpha \) and \( \beta \), but with initial conditions (i) \( x(0) = 5 \) and \( y(0) = 3.5 \), (ii) \( x(0) = 8 \) and \( y(0) = 2 \) and (iii) \( x(0) = 1 \) and \( y(0) = 1 \). What happens to the periods and amplitudes of the solutions as the initial conditions change?

(e) Plot the phase portraits - graphs of \((x(t), y(t))\) - for the four solutions you found above in the same figure as the direction field from (a). What do you see? How do the phase portraits change as you change the initial conditions?

(f) Try to explain the phenomenon you observed above.

**Hint:** This system has an equilibrium solution \((x^*, y^*) = (6, 4)\).

2. The second order, nonlinear differential equation

\[
\frac{d^2y}{dt^2} - \mu(1 - y^2)\frac{dy}{dt} + y = 0
\]

is called Van der Pol’s equation, where \( \mu \) is a positive parameter.†

(a) Without using MATLAB, find the solution to Van der Pol’s equation when \( \mu = 0 \) and \( y(0) = 2 \) and \( y'(0) = 0 \).

(b) Use MATLAB to compute (approximate) solutions to Van der Pol’s equation for \( \mu = 0.5, \mu = 1, \mu = 1.5, \mu = 2, \mu = 2.5 \) and \( \mu = 3 \) on the interval \([0, 20]\), all with initial values \( y(0) = 2 \) and \( y'(0) = 0 \).

(c) Plot the graphs of solutions that you found in (a) and (b) in the same figure, using a different color for each of the seven values of \( \mu \).

(d) What do you observe about the periods of the solutions as \( \mu \) increases? What do you observe about their amplitudes?

†The solutions are called Van der Pol oscillators, or relaxation-oscillators. Balthasar van der Pol originally proposed these differential equations to model certain electrical circuits. The blinking turn signals in cars are based on examples of such relaxation-oscillators.