Final Review Questions - Solutions

(1) A body of mass \( m = 4 \text{ lb} \) is attached to a spring-dashpot system, for which the spring constant is \( k = 1.25 \text{ lb}_\text{in}^{-1} \) and the damping coefficient is \( \gamma = 2 \text{ lb}_\text{in} \cdot \text{sec}^{-1} \).

a. Find the function \( h_1(t) \) that describes the motion of the body if it is displaced downwards from its equilibrium position by 5 inches and released with no additional velocity.

Solution: The initial value problem that describes the motion of the body is

\[
4u'' + 2u' + 1.25u = 0, \quad u(0) = 5, \quad u'(0) = 0.
\]

The characteristic equation and characteristic roots of the differential equation above are

\[
4r^2 + 2r + 1.25 = 0 \implies r_1 = -\frac{1}{4} + \frac{i}{2} \quad \text{and} \quad r_2 = -\frac{1}{4} - \frac{i}{2}
\]

The general solution of the differential equation is therefore

\[
u(t) = e^{-t/4} \left( c_1 \cos \frac{t}{2} + c_2 \sin \frac{t}{2} \right),
\]

and its derivative is

\[
u'(t) = -\frac{1}{4} e^{-t/4} \left( c_1 \cos \frac{t}{2} + c_2 \sin \frac{t}{2} \right) + \frac{1}{2} e^{-t/4} \left( -c_1 \sin \frac{t}{2} + c_2 \cos \frac{t}{2} \right).
\]

From the initial conditions, we obtain the values of \( c_1 \) and \( c_2 \):

\[
u(0) = 5 \implies c_1 = 5
\]

and

\[
u'(0) = 0 \implies -\frac{1}{4} c_1 + \frac{1}{2} c_2 = 0 \implies c_2 = \frac{c_1}{2} = \frac{5}{2}.
\]

I.e., the motion of the body is described by the function

\[
h_1(t) = e^{-t/4} \left( 5 \cos \frac{t}{2} + 2.5 \sin \frac{t}{2} \right) = \frac{\sqrt{125}}{2} e^{-t/4} \cos \left( \frac{t}{2} - \delta \right),
\]

where \( \delta = \tan^{-1}(0.5) \approx 0.4636 \).

b. Find the function \( h_2(t) \) that describes the motion of the body if, beginning from rest at equilibrium, it is acted on by an external force \( F(t) = 4 \sin(t/2) \text{ lb}_\text{in} \cdot \text{sec}^{-2} \). What is the amplitude of the steady-state solution?

The function \( h_2 \) that we seek is the solution of the initial value problem

\[
4u'' + 2u' + 1.25u = 4 \sin(t/2), \quad u(0) = 0, \quad u'(0) = 0.
\]

We found the general solution of the corresponding homogeneous problem above, so all we need now is a particular solution of the inhomogeneous problem. Using the method of undetermined coefficients, we assume that

\[
u_p = A \cos \frac{t}{2} + B \sin \frac{t}{2},
\]
so that
\[ u_p' = -A/2 \sin t/2 + B/2 \cos t/2 \quad \text{and} \quad u_p'' = -A/4 \cos t/2 - B/4 \sin t/2. \]

Plugging \( u_p, u_p' \) and \( u_p'' \) into the differential equation \( 4u'' + 2u' + 1.25u = 4 \sin t/2 \) gives
\[ 4(-A/4 \cos t/2 - B/4 \sin t/2) + 2(-A/2 \sin t/2 + B/2 \cos t/2) + 1.25(A \cos t/2 + B \sin t/2) = 4 \sin t/2 \]

\[ \implies (0.25A + B) \cos t/2 + (-A + 0.25B) \sin t/2 = 4 \sin t/2 \]

\[ \implies \begin{cases} 0.25A + B = 0 \\ -A + 0.25B = 4 \end{cases} \implies \begin{cases} A = \frac{-4}{17/16} = \frac{-64}{17} \\ B = \frac{1}{17/16} = \frac{16}{17} \end{cases} \]

Thus, the general solution of the inhomogeneous differential equation is
\[ u = e^{-t/4} (c_1 \cos t/2 + c_2 \sin t/2) - \frac{64}{17} \cos t/2 + \frac{16}{17} \sin t/2 \]

and the derivative of the general solution is
\[ u' = -\frac{1}{4} e^{-t/4} (c_1 \cos t/2 + c_2 \sin t/2) + \frac{1}{2} e^{-t/4} (-c_1 \sin t/2 + c_2 \cos t/2) + \frac{32}{17} \sin t/2 + \frac{8}{17} \cos t/2. \]

The initial conditions \( u(0) = 0 = u'(0) \) lead to the pair of equations
\[ \begin{align*}
  c_1 - \frac{64}{17} &= 0 \\
  -\frac{1}{4} c_1 + \frac{1}{2} c_2 + \frac{8}{17} &= 0 
\end{align*} \]

with solution
\[ c_1 = \frac{64}{17} \quad \text{and} \quad c_2 = \frac{16}{17}. \]

In other words, the function we seek is
\[ h_2(t) = e^{-t/4} \left( \frac{64}{17} \cos t/2 + \frac{16}{17} \sin t/2 \right) - \frac{64}{17} \cos t/2 + \frac{16}{17} \sin t/2. \]

The steady state portion of the solution is \( u_p \), and the amplitude of \( u_p \) is
\[ \sqrt{(64/17)^2 + (16/17)^2} = \frac{16}{\sqrt{17}} \approx 3.88. \]

c. Suppose that the body is replaced by a body of mass \( m = \mu \), and set in motion again as in part b. by the same external force. Find the value of \( \mu \) for which the amplitude of the steady-state his greatest.

If the mass \( m = 4 \) is replaced by mass \( m = \mu \) and you redo the calculations that produce the values of \( A \) and \( B \) (near the top of the previous page), you will find that
\[ A(\mu) = -\frac{4}{(1.25 - \mu/4)^2 + 1} \quad \text{and} \quad B(\mu) = \frac{5 - \mu}{(1.25 - \mu/4)^2 + 1} \]
The amplitude of the steady state solution for mass $\mu$ is therefore

$$R(\mu) = \sqrt{A(\mu)^2 + B(\mu)^2} = \sqrt{\frac{16 + (5 - \mu)^2}{((1.25 - \mu/4)^2 + 1)^2}} = \sqrt{\frac{16(1 + (1.25 - \mu/4)^2)}{((1.25 - \mu/4)^2 + 1)^2}}$$

$$= \sqrt{\frac{16}{(1.25 - \mu/4)^2 + 1}}.$$

Therefore, to maximize $R(\mu)$, we have to minimize the denominator

$$D(\mu) = \sqrt{(1.25 - \mu/4)^2 + 1}.$$

As a function of $\mu$, $D(\mu)$ is minimized when $(1.25 - \mu/4) = 0$, which happens when $\mu^* = 5$ and gives $D^* = 1$. This means that the maximum possible amplitude of the steady-state solution (assuming that the spring and damping coefficients remain the same) is

$$R^* = \frac{4}{1} = 4.$$

(2)

a. Find the general solution of the (third-order linear) differential equation

$$y''' - y'' - y' + y = \cos t$$

The characteristic equation for the corresponding homogeneous equation

$$y''' - y'' - y' + y = 0$$

is

$$r^3 - r^2 - r + 1 = 0 \implies r^2(r - 1) - (r - 1) = 0 \implies (r^2 - 1)(r - 1) = 0.$$

This means that the characteristic roots are $r_1 = 1$ (repeated twice) and $r_2 = -1$. It follows that the general solution of the homogeneous equation

$$y_h = e^t(c_1 + c_2t) + c_3e^{-t}.$$

Next, to find a particular solution of the inhomogeneous equation, we use the method of undetermined coefficients, and set $y_p = A\cos t + B\sin t$, in which case

$$y_p' = -A\sin t + B\cos t, \quad y_p'' = -A\cos t - B\sin t \quad \text{and} \quad y_p''' = A\sin t - B\cos t.$$

Plugging $y_p, y_p', y_p''$ and $y_p'''$ into the original differential equation gives

$$(A\sin t - B\cos t) - (-A\cos t - B\sin t) - (-A\sin t + B\cos t) + (A\cos t + B\sin t) = \cos t$$

Collecting $\cos t$ and $\sin t$ terms, we find that

$$(2A - 2B)\cos t + (2A + 2B)\sin t = \cos t \implies \begin{cases} 2A - 2B = 1 \\ 2A + 2B = 0 \end{cases}$$

From which it follows that $A = \frac{1}{4}$ and $B = -\frac{1}{4}$. Putting all of this together, we find that the general solution of the inhomogeneous equation is

$$y = y_h + y_p = e^t(c_1 + c_2t) + c_3e^{-t} + \frac{1}{4}\cos t - \frac{1}{4}\sin t.$$
b. Find the solution of this differential equation that satisfies the initial conditions

\[ y(0) = 1, \quad y'(0) = -1 \text{ and } y''(0) = 1. \]

Differentiating the general solution, above, twice gives

\[ y' = e^t(c_1 + c_2 + c_2t) - c_3e^{-t} - \frac{1}{4} \sin t - \frac{1}{4} \cos t \quad \text{and} \quad y'' = e^t(c_1 + 2c_2 + c_2t) + c_3e^{-t} - \frac{1}{4} \cos t + \frac{1}{4} \sin t. \]

The initial conditions then lead to the following system of equations for the constants \( c_1, c_2 \) and \( c_3 \):

\[
\begin{align*}
y(0) = 1 & \implies c_1 + c_3 = \frac{3}{4} \\
y'(0) = -1 & \implies c_1 + c_2 - c_3 = -\frac{3}{4} \\
y''(0) = 1 & \implies c_1 + 2c_2 + c_3 = \frac{5}{4}
\end{align*}
\]

Solving this system gives \( c_1 = -\frac{1}{8}, \ c_2 = \frac{1}{4} \) and \( c_3 = \frac{7}{8} \), so the solution that we seek is

\[ y = e^t \left( \frac{1}{4} t - \frac{1}{8} \right) + \frac{7}{8} e^{-t} + \frac{1}{4} \cos t - \frac{1}{4} \sin t. \]

(3) Use the Laplace transform to solve the initial value problems below

a. \( y'' - 2y' - 3y = 0 \quad y(0) = 0, \quad y'(0) = 1. \)

Applying the Laplace transform to both sides of the equation and inserting the initial conditions, we have

\[ \mathcal{L}(y'' - 2y' - 3y) = \mathcal{L}(0) \implies s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) - 3Y(s) = 0 \]
\[ \implies (s^2 - 2s - 3)Y(s) = 1 \quad \text{(since } y(0) = 0 \text{ and } y'(0) = 1) \]
\[ \implies Y(s) = \frac{1}{s^2 - 2s - 3} \]

The function we seek is the inverse transform of \( Y(s) \) and to find this inverse transform we simplify the expression on the right:

\[ \frac{1}{s^2 - 2s - 3} = \frac{1}{(s - 3)(s + 1)} = \frac{1/4}{s - 3} - \frac{1/4}{s + 1}, \]

using a partial fraction decomposition. It then follows from the table on page 317 (and the linearity properties of the Laplace transform) that

\[ y(t) = \mathcal{L}^{-1} \left( \frac{1/4}{s - 3} - \frac{1/4}{s + 1} \right) = \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t}. \]

b. \( y'' + 9y = \sin 2t \quad y(0) = 1, \quad y'(0) = 0. \)
Repeat the process...

\[ \mathcal{L}(y'' + 9y) = \mathcal{L}(\sin 2t) \implies s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{2}{s^2 + 4} \]

\[ \implies (s^2 + 9)Y(s) = \frac{2}{s^2 + 4} + s \]

\[ \implies Y(s) = \frac{2}{(s^2 + 9)(s^2 + 4)} + \frac{s}{s^2 + 9} \]

Partial fraction decomposition

\[ \implies Y(s) = \frac{2/5}{s^2 + 4} - \frac{2/5}{s^2 + 9} + \frac{s}{s^2 + 9} \]

Adjust constants to match Laplace transforms

\[ \implies Y(s) = \frac{1}{5} \cdot \frac{2}{s^2 + 4} - \frac{2}{15} \cdot \frac{3}{s^2 + 9} + \frac{s}{s^2 + 9} \]

It now follows that

\[ y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{5} \sin 2t - \frac{2}{15} \sin 3t + \cos 3t \]

c. \( y'' + 4y' + 5y = \varphi(t) \quad y(0) = 1, \quad y'(0) = 1, \) where

\[ \varphi(t) = \begin{cases} 
0 & : 0 \leq t < 1 \\
1 & : 1 \leq t < 3 \\
0 & : 3 \leq t
\end{cases} \]

**Note:** \( \varphi(t) = u_1(t) - u_3(t), \) where \( u_c(t) \) is unit step function — see section 6.3.

Once more unto the breach...

\[ \mathcal{L}(y'' + 4y' + 5y) = 2\mathcal{L}(u_1(t) - u_3(t)) = \mathcal{L}(u_1(t)) - \mathcal{L}(u_3(t)) \]

\[ \implies s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 5Y(s) = \frac{e^{-s} - e^{-3s}}{s} \]

\[ \implies Y(s)(s^2 + 4s + 5) = s + 5 + \frac{e^{-s}}{s} - \frac{e^{-3s}}{s} \]

\[ \implies Y(s) = \frac{s + 5}{s^2 + 4s + 5} + \frac{e^{-s}}{s(s^2 + 4s + 5)} - \frac{e^{-3s}}{s(s^2 + 4s + 5)} \]

Finding the inverse Laplace transform of the expression on the right requires a few simple steps.

**Step 1:** \( s^2 + 4s + 5 = (s + 2)^2 + 1 \) and \( \frac{s + 5}{s^2 + 4s + 5} = \frac{s + 2}{(s + 2)^2 + 1} + \frac{3}{(s + 2)^2 + 1}, \) so

\[ \mathcal{L}^{-1}\left( \frac{s + 5}{s^2 + 4s + 5} \right) = \mathcal{L}^{-1}\left( \frac{s + 2}{(s + 2)^2 + 1} \right) + \mathcal{L}^{-1}\left( \frac{3}{(s + 2)^2 + 1} \right) = e^{-2t} \cos t + 3e^{-2t} \sin t, \]

using formulas 9. and 10. from the table of Laplace transforms.

**Step 2:** Partial fraction decomposition and some rearranging gives

\[
\frac{1}{s(s^2 + 4s + 5)} = \frac{1/5}{s} - \frac{s/5}{s^2 + 4s + 5} - \frac{4/5}{s^2 + 4s + 5} = \frac{1}{5} \left( \frac{1}{s} - \frac{s}{(s + 2)^2 + 1} - \frac{4}{(s + 2)^2 + 1} \right) = \frac{1}{5} \left( \frac{1}{s} - \frac{s + 2}{(s + 2)^2 + 1} - \frac{2}{(s + 2)^2 + 1} \right)
\]
and formulas 1., 9. and 10. from the table then show that
\[
\mathcal{L}^{-1} \left( \frac{1}{s(s^2 + 4s + 5)} \right) = \frac{1}{5} \left( 1 - e^{-2t} \cos t - 2e^{-2t} \sin t \right).
\]

**Step 3.** Formula 13. from the table now gives
\[
\mathcal{L}^{-1} \left( \frac{e^{-s}}{s(s^2 + 4s + 5)} \right) = \frac{1}{5} u_1(t) \left( 1 - e^{-2(t-1)} \cos(t - 1) - 2e^{-2(t-1)} \sin(t - 1) \right)
\]
and
\[
\mathcal{L}^{-1} \left( \frac{e^{-3s}}{s(s^2 + 4s + 5)} \right) = \frac{1}{5} u_3(t) \left( 1 - e^{-2(t-3)} \cos(t - 3) - 2e^{-2(t-3)} \sin(t - 3) \right)
\]

**Finally:** Combining the results of the previous steps, the solution is
\[
y(t) = \mathcal{L}^{-1}(Y(s)) = e^{-2t} \cos t + 3e^{-2t} \sin t + \frac{1}{5} u_1(t) \left( 1 - e^{-2(t-1)} \cos(t - 1) - 2e^{-2(t-1)} \sin(t - 1) \right)
- \frac{1}{5} u_3(t) \left( 1 - e^{-2(t-3)} \cos(t - 3) - 2e^{-2(t-3)} \sin(t - 3) \right)
\]

While not pretty, the formula above can be used to evaluate \(y(t)\) for any value of \(t\) that we might want. The graph of the solution appears below.

![Graph of the solution to Problem 3(c).](image)

(4) Find the functions \(x_1(t)\) and \(x_2(t)\) that satisfy
\[
\begin{align*}
x'_1 &= 3x_1 - 2x_2 \\
x'_2 &= -x_1 + 2x_2
\end{align*}
\]
and the initial conditions
\[
x_1(0) = 1 \quad \text{and} \quad x_2(0) = -1
\]
The characteristic equation of the matrix \( A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \) is

\[
\det(A - rI) = 0 \implies \begin{vmatrix} 3 - r & -2 \\ -1 & 2 - r \end{vmatrix} = 0 \implies r^2 - 5r + 4 = 0
\]

so the eigenvalues of \( A \) are \( r_1 = 1 \) and \( r_2 = 4 \).

An eigenvector \( \xi_1 \) for the eigenvalue \( r_1 = 1 \) is found by finding a solution of \((A - I)x = 0\)

\[
\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies u = v \implies \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Likewise, an eigenvector \( \xi_2 \) for the eigenvalue \( r_2 = 4 \) is found by finding a solution of \((A - 4I)x = 0\)

\[
\begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies u = -2v \implies \xi_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.
\]

Therefore, the general solution of the system above is given by

\[
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.
\]

The condition \( \bar{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) leads to a pair of equations for \( c_1 \) and \( c_2 \):

\[
c_1 - 2c_2 = 1 \\
c_1 + c_2 = -1
\]

\( \implies \) \( c_1 = -\frac{1}{3} \) and \( c_2 = -\frac{2}{3} \),

So the functions we seek are

\[
x_1(t) = -\frac{1}{3} e^t + \frac{4}{3} e^{4t} \quad \text{and} \quad x_2(t) = -\frac{1}{3} e^t - \frac{2}{3} e^{4t}
\]

(5) Find the general solution of the system

\[
\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 2t + 1 \\ t - 5 \end{bmatrix}
\]

and describe the long-term behavior of all solutions of this system.

The eigenvalues of the matrix \( A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \) are \( r_1 = -1 \) and \( r_2 = -4 \), as you should verify. Eigenvectors for these eigenvalues are

\[
\eta_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } r_1 = -1 \quad \text{and} \quad \eta_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ for } r_2 = -4,
\]

as you should also verify. This means that the general solution of the homogeneous system

\[
\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]
is given by
\[ x_h = c_1e^{-t}\eta_1 + c_2e^{-4t}\eta_2 = \begin{bmatrix} c_1e^{-t} - 2c_2e^{-4t} \\ c_1e^{-t} + c_2e^{-4t} \end{bmatrix} \]

Because the nonhomogeneous term is the linear (vector-valued) function
\[ f(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \]

the method of undetermined coefficients indicates that there is a (different) linear function that is a particular solution of the system. In other words, we expect the particular solution to look like this
\[ x_p = \bar{a}t + \bar{b}, \]

where \( \bar{a} \) and \( \bar{b} \) are constant vectors. Plugging \( x_p \) in the differential equation gives
\[ x'_p = A x_p + f(t) \iff \bar{a} = A(\bar{a}t + \bar{b}) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \]

since \( x'_p = \bar{a} \). Rearranging a little gives the equation
\[ (A\bar{a})t + (A\bar{b} - \bar{a}) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \]

and equating the \( t \)-coefficients and the constant coefficients on both sides yields the pair of equations
\[ A\bar{a} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad \text{and} \quad (A\bar{b} - \bar{a}) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}. \]

Solving the left-hand equation first, we have
\[ \bar{a} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/4 \end{bmatrix}. \]

Inserting this into the second equation above gives \( A\bar{b} = \bar{a} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} \), so
\[ \bar{b} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}^{-1} \left( \begin{bmatrix} 3/2 \\ 5/4 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right) = \frac{1}{4} \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 25/4 \end{bmatrix} = -\begin{bmatrix} 27/8 \\ 77/16 \end{bmatrix}. \]

Therefore the particular solution we seek is
\[ x_p = \begin{bmatrix} 3/2 \\ 5/4 \end{bmatrix} t - \begin{bmatrix} 27/8 \\ 77/16 \end{bmatrix} \]

and the general solution of the nonhomogeneous system is
\[ x = x_h + x_p = \begin{bmatrix} c_1e^{-t} - 2c_2e^{-4t} \\ c_1e^{-t} + c_2e^{-4t} \end{bmatrix} + \begin{bmatrix} 3t/2 - 27/8 \\ 5t/4 - 77/16 \end{bmatrix}. \]
Now, regardless of the values of $c_1$ and $c_2$, if $t$ is sufficient large, then

$$\Phi_h(t) \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and therefore every solution of the system satisfies

$$\Phi = \Phi_h + \Phi_p \approx \Phi_p = \begin{bmatrix} 3/2 \\ 5/4 \end{bmatrix} t - \begin{bmatrix} 27/8 \\ 77/16 \end{bmatrix},$$

for $t$ sufficiently large. In other words, if $\Phi(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ is any solution of the system, then

$$u(t) \approx \frac{3}{2} t - \frac{27}{8} \quad \text{and} \quad v(t) \approx \frac{5}{4} t - \frac{77}{16}$$

when $t$ is sufficiently large.

This means that the trajectories of all solutions of this system approach a fixed line in the phase plane (the $(u, v)$-plane) as $t \to \infty$. To find this line, we observe that if $u = \frac{3}{2} t - \frac{27}{8}$, then $t = \frac{2}{3} u + \frac{9}{4}$, and therefore

$$v = \frac{5}{4} t - \frac{77}{16} = \frac{5}{4} \left( \frac{2}{3} u + \frac{9}{4} \right) - \frac{77}{16} = \frac{5}{6} u - 2$$

is the equation of the line that we seek. Some typical trajectories are plotted below, as well as the line they are all approaching.

![Figure 2: Trajectories for the system in problem 5.](image-url)
Two tanks, A and B, are connected to each other and to the outside by pipes. Initially, tank A contains 200 liters of pure water and tank B contains 300 liters of pure water. At time $t = 0$, saline solution begins to flow in, out and between the two tanks, as described below.

- Solution with a concentration of 20 gms/liter flows into Tank A at a rate of 8 liters/min.
- Solution with a concentration of 10 gms/liter flows into Tank B at a rate of 18 liters/min.
- Well mixed solution flows from Tank A to Tank B at a rate of 6 liters/min.
- Well mixed solution flows from Tank B to Tank A at a rate of 6 liters/min.
- Well mixed solution flows from Tank A to the outside at a rate of 8 liters/min.
- Well mixed solution flows from Tank B to the outside at a rate of 18 liters/min.

Find the functions $Q_A(t)$ and $Q_B(t)$ that give the quantities of salt in tanks A and B at time $t$, respectively. Approximately how much salt is in each tank when $t$ is large? Be precise.

To begin, you might want to sketch a diagram (as above) to make the information easier to digest.

Next, find the rates in and rates out (in gms/min) for each tank:

**Tank A:**

rate in = $20 \text{ (gms/liter)} \times 8 \text{ (liter/min)} + \frac{Q_B}{300} \text{ (gms/liter)} \times 6 \text{ (liter/min)} = \left(160 + \frac{1}{50}Q_B\right) \text{ (gms/min)}$
rate out = \frac{Q_A}{200} (gms/liter) \times 14(liter/min) = \frac{7}{100} Q_A(gms/min)

Tank B:
rate in = 10(gms/liter) \times 18(liter/min) + \frac{Q_A}{200} (gms/liter) \times 6(liter/min) = \left(180 + \frac{3}{100} Q_A\right) (gms/min)
rate out = \frac{Q_B}{300} (gms/liter) \times 24(liter/min) = \frac{8}{100} Q_B(gms/min)

Now, translate this information into a pair of differential equations for \(Q_A\) and \(Q_B\):

\begin{align*}
Q'_A &= -0.07 Q_A + 0.02 Q_B + 160 \\
Q'_B &= 0.03 Q_A - 0.08 Q_B + 180
\end{align*}

The matrix for this system is

\[ M = \begin{bmatrix} -0.07 & 0.02 \\ 0.03 & -0.08 \end{bmatrix} \]

The eigenvalues of \(M\) are \(\lambda_1 = -0.1\) and \(\lambda_2 = -0.05\), as you should check, and corresponding eigenvectors are

\[ \bar{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \text{ (for } \lambda_1) \text{ and } \bar{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ (for } \lambda_2), \]

as you should also check.

Thus the general solution of the homogeneous system \(\bar{Q}' = M \bar{Q}\) is given by

\[ \bar{Q}_h = c_1 e^{-0.1t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 e^{-0.05t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Next, to find a particular solution of the original non homogeneous system, I’ll use the method of undetermined coefficients (as in the previous problem) which indicates that there is a constant vector solution, \(\bar{Q}_p\). In this case we have \(\bar{Q}'_p = \bar{u}\), so we have to solve the linear system

\[ \bar{Q}'_p = M \bar{Q}_p + \begin{bmatrix} 160 \\ 180 \end{bmatrix} \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.07 & 0.02 \\ 0.03 & -0.08 \end{bmatrix} \bar{Q}_p + \begin{bmatrix} 160 \\ 180 \end{bmatrix} \]

This implies that

\[ \bar{Q}_p = -\begin{bmatrix} -0.07 & 0.02 \\ 0.03 & -0.08 \end{bmatrix}^{-1} \begin{bmatrix} 160 \\ 180 \end{bmatrix} = -\frac{1}{0.005} \begin{bmatrix} -0.08 & -0.02 \\ -0.03 & -0.07 \end{bmatrix} \begin{bmatrix} 160 \\ 180 \end{bmatrix} = \frac{1}{0.005} \begin{bmatrix} 16.4 \\ 17.4 \end{bmatrix} = \begin{bmatrix} 3280 \\ 3480 \end{bmatrix} \]

The general solution of the non homogeneous problem is therefore

\[ \bar{Q} = \bar{Q}_h + \bar{Q}_p = c_1 e^{-0.1t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 e^{-0.05t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3280 \\ 3480 \end{bmatrix}. \]
Given that the tanks initially contained pure water, it follows that $Q_A(0) = 0 = Q_B(0)$, so that $\mathbf{Q}(0) = \mathbf{0}$, and we can use this to find the coefficients $c_1$ and $c_2$ above. Specifically, we must have

$$\mathbf{0} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \mathbf{Q}_p \Rightarrow c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3280 \\ 3480 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3280 \\ 3480 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/2 \end{bmatrix} \begin{bmatrix} 3280 \\ 3480 \end{bmatrix} = \begin{bmatrix} 40 \\ -3360 \end{bmatrix}$$

Thus the solution of the initial value problem in vector form is

$$\begin{bmatrix} Q_A \\ Q_B \end{bmatrix} = 40e^{-0.1t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} - 3360e^{-0.05t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3280 \\ 3480 \end{bmatrix}.$$