One of the simplest models for the interaction of predator and prey species is given by the Lotka-Volterra equations:

\[
\frac{dx}{dt} = r_1 x - \alpha xy \\
\frac{dy}{dt} = \beta xy - r_2 y
\]

(a) Plot a direction field for the Lotka-Volterra system with $r_1 = 0.4$, $r_2 = 0.3$, $\alpha = 0.1$ and $\beta = 0.05$. You should plot the direction field in the first quadrant, with $0 < x < 10$ and $0 < y < 10$.

MATLAB Commands for (a):

\[
\begin{align*}
>> & [x y] = meshgrid(0:0.25:10, 0:0.25:10); \\
>> & dx = 0.4 \times x - 0.1 \times x \times y; \\
>> & dy = 0.05 \times x \times y - 0.3 \times y; \\
>> & L = sqrt(dx.^2 + dy.^2); \\
>> & quiver(x, y, dx./L, dy./L, 0.6); axis tight
\end{align*}
\]

(b) Use ode45 to compute the solution to Lotka-Volterra system in the interval $[0, 40]$ when $r_1 = 0.4$, $r_2 = 0.3$, $\alpha = 0.1$ and $\beta = 0.05$, assuming that $x(0) = 10$ and $y(0) = 5$.

MATLAB Commands for (b):

\[
\begin{align*}
>> & PP = @(t, y) [0.4 \times y(1) - 0.1 \times y(1) \times y(2); 0.05 \times y(1) \times y(2) - 0.3 \times y(2)]; \\
>> & [t y] = ode45(PP, [0 40], [10; 5]);
\end{align*}
\]

(c) Plot the graphs of the solutions $x(t)$ and $y(t)$ in the same figure (using different colors), and describe how the populations evolve over time. What are the periods and amplitudes of populations?

Commands for (c):

\[
\begin{align*}
>> & plot(t, y(:, 1), t, y(:, 2), 'r');
\end{align*}
\]
Output:

Plots of $x(t)$ and $y(t)$ when $x(0) = 10$ and $y(0) = 5$.

The periods are both about 19. The amplitude for the prey population $x(t)$ (blue) is about 7.5 and the amplitude of the predator population $y(t)$ (red) is about 4.3.

(d) Repeat (b) and (c) with the same values of $r_1, r_2, \alpha$ and $\beta$, but with initial conditions (i) $x(0) = 5$ and $y(0) = 3.5$, (ii) $x(0) = 8$ and $y(0) = 2$ and (iii) $x(0) = 1$ and $y(0) = 1$. What happens to the periods and amplitudes of the solutions as the initial conditions change?

MATLAB Commands and plots:

```matlab
>> [t y] = ode45(PP, [0 40], [5;3.5]);
>> plot(t,y(:,1),t,y(:,2),'r');
```

Plots of $x(t)$ (blue) and $y(t)$ (red), when $x(0) = 5$ and $y(0) = 3.5$. 

```matlab
>> plot(t,y(:,1),t,y(:,2),'r');
```
\[ [t \; y] = \text{ode45}(PP, [0 \; 40], [8; 2]); \]
\[ \text{plot}(t, y(:,1), t, y(:,2), 'r'); \]

Plots of \( x(t) \) (blue) and \( y(t) \) (red), when \( x(0) = 8 \) and \( y(0) = 2 \).

\[ [t \; y] = \text{ode45}(PP, [0 \; 40], [1; 1]); \]
\[ \text{plot}(t, y(:,1), t, y(:,2), 'r'); \]

Plots of \( x(t) \) (blue) and \( y(t) \) (red), when \( x(0) = 1 \) and \( y(0) = 1 \).

The periods, in all three cases appear to be about the same, but the amplitudes for both populations change with the initial conditions. The smallest amplitudes occur for \( x(0) = 5 \) and \( y(0) = 3.5 \) – about 2.8 for the prey population and about 1.6 for the predator.
population; and the largest amplitudes occur when \( x(0) = 1 \) and \( y(0) = 1 \) – about 25 for the prey population and about 14 for the predator population.

(e) Plot the phase portraits – graphs of \((x(t), y(t))\) – for the four solutions you found above in the same figure. What do you see? How do the phase portraits change as you change the initial conditions?

**MATLAB Commands (for phase portraits):**

```matlab
>> [t y]=ode45(PP,[0 40],[10;5]);
>> plot(y(:,1),y(:,2));
>> hold on
>> [t y]=ode45(PP,[0 40],[5;3.5]);
>> plot(y(:,1),y(:,2),'r');
>> [t y]=ode45(PP,[0 40],[8;2]);
>> plot(y(:,1),y(:,2),'g');
>> [t y]=ode45(PP,[0 40],[1;1]);
>> plot(y(:,1),y(:,2),'k');
```

**Output:**

![Plot of four phase portraits](image)

Plot of four phase portraits, corresponding to initial conditions \( x(0) = 10 \) and \( y(0) = 5 \) (blue); \( x(0) = 5 \) and \( y(0) = 3.5 \) (red); \( x(0) = 8 \) and \( y(0) = 2 \) (green); \( x(0) = 1 \) and \( y(0) = 1 \) (black)

(f) Try to explain the phenomenon you observed above.

The system has an equilibrium solution at \((6, 4)\) – marked by the black ‘*’ in the phase portrait plot above – and it appears that the farther the point \((x(0), y(0))\) is from this point, the larger the diameter of the phase portrait, corresponding to bigger amplitudes in the periodic behavior of the populations.

2. The second order, nonlinear differential equation

\[
\frac{d^2y}{dt^2} - \mu(1-y^2)\frac{dy}{dt} + y = 0
\]

is called Van der Pol’s equation, where \( \mu \) is a positive parameter.†

†The solutions are called Van der Pol oscillators.
(a) Without using MATLAB, find the solution to Van der Pol’s equation when $\mu = 0$ and $y(0) = 2$ and $y'(0) = 0$.

$$\frac{d^2y}{dt^2} + y = 0 \implies y = c_1 \cos t + c_2 \sin t$$

and the initial conditions $y(0) = 2$ and $y'(0) = 0$ imply that $c_1 = 2$ and $c_2 = 0$, so the solution in this case is

$$y(t) = 2 \cos t.$$

(b) Use MATLAB to compute (approximate) solutions to Van der Pol’s equation for $\mu = 0.5$, $\mu = 1$, $\mu = 1.5$, $\mu = 2$, $\mu = 2.5$ and $\mu = 3$ on the interval $[0, 20]$, all with initial values $y(0) = 2$ and $y'(0) = 0$.

*In order to use MATLAB to approximate the solutions we want, we need to convert the second order Van der Pol equation into a pair of first order equations. We do this by setting $y_1 = y$ and $y_2 = y' = y'_1$. With this, the Van der Pol equation becomes the pair of equations

$$y'_1 = y_2$$
$$y'_2 = \mu(1 - y_1^2)y_2 - y_1$$

MATLAB commands follow below.

(c) Plot the graphs of solutions that you found in (a) and (b) in the same figure, using a different color for each of the seven values of $\mu$.

**MATLAB Commands for 2b and 2c:**

```matlab
>> t=[0:0.05:20];
>> plot(t,2*cos(t),'k')
>> hold on
>> VdP=@(t,y) [y(2); 0.5*(1-y(1).^2).*y(2)-y(1)];
>> [t y]=ode45(VdP, [0 20], [2;0]);
>> plot(t,y(:,1));
>> VdP=@(t,y) [y(2); (1-y(1).^2).*y(2)-y(1)];
>> [t y]=ode45(VdP, [0 20], [2;0]);
>> plot(t,y(:,1),'r');
>> VdP=@(t,y) [y(2); 1.5*(1-y(1).^2).*y(2)-y(1)];
>> [t y]=ode45(VdP, [0 20], [2;0]);
>> plot(t,y(:,1),'g');
>> VdP=@(t,y) [y(2); 2*(1-y(1).^2).*y(2)-y(1)];
>> [t y]=ode45(VdP, [0 20], [2;0]);
>> plot(t,y(:,1),'y');
>> VdP=@(t,y) [y(2); 2.5*(1-y(1).^2).*y(2)-y(1)];
>> [t y]=ode45(VdP, [0 20], [2;0]);
>> plot(t,y(:,1),'c');
>> VdP=@(t,y) [y(2); 3*(1-y(1).^2).*y(2)-y(1)];
>> [t y]=ode45(VdP, [0 20], [2;0]);
>> plot(t,y(:,1),'m');
```
Plots for $\mu = 0$ (black); $\mu = 0.5$ (blue); $\mu = 1$ (red); $\mu = 1.5$ (green); $\mu = 2$ (yellow); $\mu = 2.5$ (cyan); and $\mu = 3$ (magenta).

(d) What do you observe about the periods of the solutions as $\mu$ increases? What do you observe about their amplitudes?

The amplitudes all remain exactly the same (4), but the periods appear to be growing with $\mu$. 