Midterm Exam
The midterm is closed-book, you are only allowed to use one page of notes and a calculator. Please attach your formula sheet.

There are 6 problems, 100 points total. The point values for each problem are listed below. Some problems have multiple parts. Please provide a detailed calculation (not just a number) or an explanation (or both, as needed) to support your idea of the right answer in each problem. If you run out of space in any of the questions please use the last page and indicate that you are doing so.

REMEMBER: THIS TEST IS TO BE ENTIRELY YOUR OWN EFFORTS. Cheating on this test will result in the severest disciplinary action possible under UCSC rules.

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1. (10 points) Suppose that 10 people are seated in a random manner in a row of 10 theater seats. What is the probability that two particular people $A$ and $B$ will be seated next to each other? Justify your answer.

**SOLUTION:** There are $\binom{10}{2}$ ways to seat $A$ and $B$ on ten seats, without keeping track of order (i.e. we do not take into account who we seat first). There are 9 ways to seat $A$ and $B$ next to each other, again without taking order into account. Hence, the probability that $A$ and $B$ sit next to each other is

$$\frac{9}{\binom{10}{2}} = \frac{1}{5}$$

2. (15 points) Suppose that $A$, $B$, and $C$ are three independent events such that $Pr(A) = 1/4$, $Pr(B) = 1/3$, and $Pr(C) = 1/2$. Determine the probability that exactly one of these events will occur.

**SOLUTION:** The event that exactly one of the events will occur can be represented as

$$\left( A \cap \bar{B} \cap \bar{C} \right) \cup \left( \bar{A} \cap B \cap \bar{C} \right) \cap \left( \bar{A} \cap \bar{B} \cap C \right).$$

By construction, these events are disjoint, meaning that the probability of their union equals to the sum of their probabilities. Further recall that events $A$, $B$, and $C$ are independent. Thus,

$$P\left( A \cap \bar{B} \cap \bar{C} \right) = P(A)P(\bar{B})P(\bar{C}) = \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{12}$$

$$P\left( \bar{A} \cap B \cap \bar{C} \right) = P(\bar{A})P(B)P(\bar{C}) = \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{8}$$

$$P\left( \bar{A} \cap \bar{B} \cap C \right) = P(\bar{A})P(\bar{B})P(C) = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}$$

Therefore,

$$P( \text{ exactly one event } ) = \frac{1}{12} + \frac{1}{8} + \frac{1}{2} = \frac{11}{24} = 0.458$$
3. (20 points) A plant pathologist studying a bacterial disease of apple trees surveyed a large population of commercial apple trees of a variety called Empire. He noted that 40% of all the trees had red flowers, while the remainder had white flowers. In addition, 30% of the trees showed some evidence of being infected with the bacterial disease and 10% of the trees were white-flowered and infected.

(a) (5 points) What percentage of the trees were red-flowered and infected?

**SOLUTION:** Let the events be defined as follows:

- \( R = \) red flowers
- \( \bar{R} = W = \) white flowers
- \( B = \) bacterial disease

The problem statement gives us the probabilities of these events as

\[
\begin{align*}
P(R) &= 0.4 & P(W) &= 0.6 \\
P(B) &= 0.3 & P(\bar{B}) &= 0.7 \\
P(B \cap W) &= 0.1
\end{align*}
\]

We seek \( P(B \cap R) \). To find this probability, we need to make use of the fact that we know \( P(B \cap W) \). If \( S \) is the entire sample space, then the equality holds:

\[
B = B \cap S = B \cap (W \cup R) = (B \cap W) \cup (B \cap R)
\]

where the last union is disjoint. Therefore,

\[
P(B) = P(B \cap W) + P(B \cap R) \\
\Rightarrow P(B \cap W) = P(B) - P(B \cap W) = 0.3 - 0.1 = 0.2
\]

(b) (5 points) What is the probability that a white-flowered tree will be infected?

**SOLUTION:** We seek \( P(B|W) \) and we employ Bayes’ Theorem:

\[
P(B|W) = \frac{P(B \cap W)}{P(W)} = \frac{0.1}{0.6}
\]
(c) (5 points) What is the probability that a red-flowered tree will be infected?

**SOLUTION:** We see $P(B|R)$.

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{0.2}{0.4} = 0.5$$

(d) (5 points) Based on your answers to the previous questions, are flower color and disease resistance independent? (Justify your answer).

**SOLUTION:**

$$P(B|R) = 0.5 \neq P(B),$$

meaning that probability of infection is dependent upon flower color.
4. (20 points) Suppose there are three boxes, $A$, $B$ and $C$, each of which contains two coins. Box $A$ has two pennies, Box $B$ one penny and one nickel, and Box $C$ two nickels. A box is chosen at random and then a coin is chosen at random from that box. The coin chosen turns out to be a nickel. What is the probability that the other coin in the chosen box is also a nickel? Show your work.

NOTE: Here “at random” means equally likely among the alternatives.

**SOLUTION:** Here we use Bayes’ Theorem. The probability that the second coin the box is a nickel, given that the first coin is a nickel, is the same probability as that the probability that the box is $C$ given that the first coin is a nickel. Letting $A, B,$ and $C$ be the events that we’ve picked boxes $A$, $B$, and $C$, respectively, and $N$ that the first coin is a nickel, we seek $P(C|N)$. By Bayes’ Theorem,

\[
P(C|N) = \frac{P(N|C)P(C)}{P(N|A)P(A) + P(N|B)P(B) + P(N|C)P(C)}
\]

\[
= \frac{1 \cdot 1/3}{1/3(0 + 1/2 + 1)} = \frac{1}{3/2} = 2/3
\]
5. (25 points) Assume the random variables $X$ and $Y$ have a p.d.f given by

$$f(x, y) = \begin{cases} \ cxy & 0 \leq x \leq 1, \ 0 \leq y \leq 1; \ 0 \leq x + y \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

with $c$ a constant.

(a) (10 points) Find the value of $c$.

**SOLUTION:** We use the fact that $\int \int f(x, y)dxdy = 1$. If we let $x$ range from 0 to 1, then $y$ is restricted between 0 and $1 - x$. Thus,

$$\int_0^1 \int_0^{1-x} cxy dydx = \frac{c}{2} \int_0^1 x y^2 |_{y=0}^{1-x} dx = \frac{c}{2} \int_0^1 x(1-x)^2 dx$$

$$= \frac{c}{2} \int_0^1 x - 2x^2 + x^3 dx = \frac{c}{2} \left[ \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1$$

$$= \frac{c}{2} \left[ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{c}{24} = 1$$

Thus, $c = 24$. 
(b) (15 points) Compute $Pr(X < Y)$.

**SOLUTION:** Consider $0 \leq x \leq 1$. Because we have the condition that $0 \leq x + y \leq 1$, if $x \geq 1/2$, then we need $y \leq x$. This means that $x \geq 1/2$ breaks the condition of $X < Y$. On the other hand, if $0 \leq x \leq 1/2$, then to satisfy $X < Y$, we require that $x \leq y \leq 1 - x$. These conditions, then, define the region of integration such that $X < Y$:

$$P(X < y) = \int_0^{1/2} \int_x^{1-x} 24xy \, dy \, dx = \int_0^{1/2} 24x \left[ \int_x^{1-x} y \, dy \right] \, dx = \int_0^{1/2} 12x \left[ y^2 \right]_x^{1-x} \, dx$$

$$= \int_0^{1/2} 12x(1 - 2x + x^2 - x^2) \, dx = 12 \int_0^{1/2} x - 2x^2 \, dx$$

$$= 12 \left[ \frac{x^2}{2} - \frac{2}{3} x^3 \right]_0^{1/2} = 12 \left[ \frac{1}{8} - \frac{1}{12} \right] = 1/2$$
6. (10 points) Suppose that events $A$ and $B$ are that people have diseases $a$ and $b$, respectively. Suppose that having either disease leads to hospitalization $H = A \cup B$. If $A$ and $B$ are believed to be independent events with $0 < Pr(A) < 1, 0 < Pr(B) < 1$ and $0 < Pr(A \cup B) < 1$, show that


Thus if hospital populations are compared, a spurious negative association between $A$ and $B$ might be found. This is called Berkson’s Paradox.

**SOLUTION:**

$$P(A|B, H) = \frac{P(A \cap B)}{P(B)} = \frac{A \cap \{B \cap (A \cup B)\}}{B \cap (A \cup B)} = P(A)$$

$$P(A|H) = P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)}$$

But $0 < P(A \cup B) < 1$, so

$$P(A) < \frac{P(A)}{P(A \cup B)}$$