1. Assume that \(X_1, \ldots, X_n\) is a random sample from a normal distribution with unknown mean \(\mu\) and known variance \(\sigma^2 = 1\). How large must the sample size \(n\) be in order for the confidence interval for \(\mu\), with confidence coefficient 0.9, to have length less than 0.1? (It is given that \(\Phi(1.645) = 0.95\), that is, the value of the standard normal distribution function at 1.645 is 0.95.)

**Solution:** The 90\% confidence interval for \(\mu\) is given by \((\bar{x}_n - (c\sigma/\sqrt{n}), \bar{x}_n + (c\sigma/\sqrt{n}))\), where \(\sigma = 1\) and \(c = 1.645\), using the information from the problem statement. Therefore, the length of the 90\% confidence interval for \(\mu\) is \(3.29/\sqrt{n}\), and for this to be less than 0.1, we must have \(n > 1082.41\), that is, the sample size must be greater than or equal to 1083.

2. Suppose that, conditional on (unknown) mean parameter \(\theta > 0\), \(X_1, \ldots, X_n\) form a random sample from a Poisson distribution with probability function

\[
f(x \mid \theta) = \frac{e^{-\theta} \theta^x}{x!}, \quad x = 0, 1, 2, \ldots
\]

(a) What is the posterior distribution for \(\theta\) under an exponential prior distribution with mean 1?

**Solution:** We have shown that the posterior distribution for \(\theta\) under a gamma(\(\alpha, \beta\)) prior is given by a gamma distribution with shape parameter \(\alpha + n\bar{x}_n\) and rate parameter \(\beta + n\).

The exponential prior is the special case of the gamma(\(\alpha, \beta\)) prior with \(\alpha = \beta = 1\), and therefore, the posterior distribution for \(\theta\) is a gamma distribution with shape parameter 1 + \(n\bar{x}_n\) and rate parameter 1 + \(n\).

(b) Consider a specific data set, with sample size \(n = 10\) and sample mean 201.8, assumed to arise as a random sample from a Poisson distribution with mean \(\theta\). Compare for this data set the M.L.E. \(\hat{\theta} = \bar{x}_n\) with the posterior expectation for \(\theta\), the latter obtained under the prior from part (a). What is the reason (or reasons) for the difference between the values of the M.L.E. and the posterior expectation?

**Solution:** The M.L.E. is \(\bar{x}_n = 201.8\), and the posterior expectation

\[
E(\theta \mid x_1, \ldots, x_n) = \frac{1 + n\bar{x}_n}{1 + n} = \frac{n^{-1} + \bar{x}_n}{n^{-1} + 1} = 183.55.
\]

One reason for the difference between the values of the M.L.E. and the posterior expectation is that the exponential prior distribution is fairly incompatible with the data; note that the prior expectation for the Poisson mean is \(\alpha/\beta = 1\), whereas the sample mean is 201.8. Another reason is the small sample size. Note that, with the same exponential prior and for data sets with the same sample mean, the posterior expectation will get closer to the M.L.E. estimate as the the sample size increases. For instance, with \(n = 100\), \(E(\theta \mid x_1, \ldots, x_n) = 199.81\), and with \(n = 1000\), \(E(\theta \mid x_1, \ldots, x_n) = 201.6\).
3. Suppose that \(X_1, \ldots, X_n\) form a random sample from the Rayleigh distribution, which is a continuous distribution with probability density function
\[
f(x \mid \theta) = \theta x \exp(-0.5\theta x^2), \quad \text{for } x > 0
\]
where \(\theta > 0\) is the (unknown) parameter of the distribution.

(a) Obtain the Fisher information \(I_n(\theta)\) in the random sample \((X_1, \ldots, X_n)\).

**Solution:** The regularity conditions are satisfied for the Rayleigh distribution (the sample space does not depend on \(\theta\)), and therefore \(I_n(\theta) = nI(\theta)\), where \(I(\theta) = \operatorname{E}(-\partial^2 \log f(x \mid \theta)/\partial \theta^2) = \operatorname{E}(1/\theta^2) = 1/\theta^2\). Hence, \(I_n(\theta) = n/\theta^2\).

(b) Derive the M.L.E. of \(\theta\). Obtain the asymptotic confidence interval, with confidence coefficient \(\gamma\), for \(\theta\) based on the large-sample normal approximation to the distribution of the maximum likelihood estimator of \(\theta\).

**Solution:** The M.L.E. was obtained as part of the first exam, \(\hat{\theta}_n = 2n/(\sum_{i=1}^n x_i^2)\). The asymptotic distribution of the M.E. estimator is normal with mean \(\theta\), and standard deviation \(1/\sqrt{I_n(\theta)}\). Using the result from part (a), and estimating \(\theta\) by the M.L. estimate \(\hat{\theta}\), the standard deviation becomes \(\sqrt{n}/\theta\). Therefore, the asymptotic confidence interval with confidence coefficient \(\gamma\) is given by \((\hat{\theta} - c(\hat{\theta}/\sqrt{n}), \hat{\theta} + c(\hat{\theta}/\sqrt{n}))\), where \(c\) is the \((1 + \gamma)/2\) percentile of the standard normal distribution, that is, \(\Phi(c) = (1 + \gamma)/2\).

4. Consider Bayesian inference for the same distribution with problem 3, that is, now conditional on \(\theta > 0\), the \(X_1, \ldots, X_n\) form a random sample from the Rayleigh distribution with probability density function \(f(x \mid \theta) = \theta x \exp(-0.5\theta x^2)\), for \(x > 0\). Consider a gamma distribution with parameters \(\alpha\) and \(\beta\) as the prior distribution for \(\theta\), that is, \(\xi(\theta) \propto \theta^{\alpha-1} \exp(-\beta\theta)\), for \(\theta > 0\).

(a) Show that the gamma distribution above provides the conjugate prior for \(\theta\), and obtain the parameters of the posterior distribution given data \(x = (x_1, \ldots, x_n)\).

**Solution:** The posterior distribution can be written proportional to
\[
\xi(\theta \mid x) \propto \theta^n \exp \left(-0.5\theta \sum_{i=1}^n x_i^2\right) \theta^{\alpha-1} \exp(-\beta\theta) \propto \theta^{(n+\alpha)-1} \exp \left(- (\beta + 0.5 \sum_{i=1}^n x_i^2) \theta \right)
\]
which can be recognized as a gamma distribution with posterior hyperparameters \(\alpha^* = \alpha + n\) and \(\beta^* = \beta + 0.5 \sum_{i=1}^n x_i^2\).

(b) Obtain the posterior expectation for the median of the Rayleigh distribution.

**Solution:** As derived for problem 3 of exam 1, the median of the Rayleigh distribution is \(\eta(\theta) = \)
\[(2 \log(2))^{1/2}/\theta^{1/2}. \text{ Therefore,}\]

\[
E(\eta(\theta) \mid x) = \frac{(2 \log(2))^{1/2}}{\Gamma(\alpha^*)} \int_0^\infty \theta^{1/2} \xi(\theta) \mid x \, d\theta
\]

\[
= \frac{(2 \log(2))^{1/2} \beta^* \alpha^*}{\Gamma(\alpha^*)} \int_0^\infty \theta(\alpha^*-0.5) \exp(-\beta^* \theta) \, d\theta
\]

\[
= \frac{(2 \log(2))^{1/2} \Gamma(n+\alpha-0.5)}{(\beta+0.5 \sum_{i=1}^n x_i^2)^{-1/2} \Gamma(\alpha^*)}
\]

where the integral in the second line is obtained through the normalizing constant of a gamma distribution with parameters \(\alpha^* - 0.5 = \alpha + n - 0.5 > 0\) and \(\beta^*\).

(c) Derive the expression for the posterior predictive density, \(f(x_{n+1} \mid x)\), corresponding to future response \(X_{n+1}\) with (unobserved) value \(x_{n+1}\).

Solution: Using the definition of the posterior predictive density,

\[
f(x_{n+1} \mid x) = \int f(x_{n+1} \mid \theta) \xi(\theta) \mid x \, d\theta
\]

\[
= \frac{\beta^* x_{n+1}^{\alpha^*}}{\Gamma(\alpha^*)} \int_0^\infty \theta^{\alpha^*+1} \exp(-0.5 \theta x_{n+1}^2) \frac{\beta^* \alpha^*}{\Gamma(\alpha^*)} \theta^{\alpha^*-1} \exp(-\beta^* \theta) \, d\theta
\]

\[
= \frac{\beta^* x_{n+1}^{\alpha^*}}{\Gamma(\alpha^*)} \frac{\beta^* + 0.5 \sum_{i=1}^n x_i^2}{\Gamma(\alpha^*)} \frac{\Gamma(\alpha^*+1)}{(\beta + 0.5 \sum_{i=1}^n x_i^2)^{\alpha^*+1}}, \text{ for } x_{n+1} > 0
\]

where the integral in the third line is evaluated using the fact that the integrand is proportional to the density of a gamma distribution with parameters \(\alpha^* + 1 = \alpha + n + 1\) and \(\beta^* + 0.5 x_{n+1}^2 = (\beta + 0.5 \sum_{i=1}^n x_i^2) + 0.5 x_{n+1}^2\).