AMS 206: Classical and Bayesian Inference (Winter 2015)

Solutions for selected homework 4 problems

1. Exercise 7.3.16
Solution: Note that, for fixed mean $\mu$, the standard deviation, $\sigma$, enters the normal likelihood through $f_n(x \mid \sigma) \propto \sigma^{-n} \exp\{-0.5 \sum_{i=1}^{n} (x_i - \mu)^2 / \sigma^2\}$. Therefore, if the prior density is of the form $\xi(\sigma) \propto \sigma^{-a} \exp(-b/\sigma^2)$, then the posterior density will be of the same form. What remains is to determine the range of values for the prior hyperparameters $a$ and $b$, and to find the normalizing constant for $\xi(\sigma)$. The normalizing constant can be obtained using the change of variables $y = \sigma^{-2}$ in the integral, and the definition of the Gamma function. Specifically,

$$
\int_{0}^{\infty} \sigma^{-a} \exp(-b/\sigma^2) \, d\sigma = \frac{1}{2} \int_{0}^{\infty} y^{(a-3)/2} \exp(-by) \, dy = \frac{\Gamma((a-1)/2)}{2b^{(a-1)/2}}
$$

provided $a > 1$ and $b > 0$.

2. Exercise 7.10.7
Solution: (a) Assuming independence for the lifetimes of the three light bulbs, the likelihood function is given by the product of the three exponential densities with means $\theta$, $2\theta$ and $3\theta$. It is straightforward to find the M.L.E. of $\theta$ by differentiating the likelihood resulting in $\hat{\theta} = \{X_1 + (X_2/2) + (X_3/3)\}/3$.

(b) Again, assuming independence (conditionally on the parameter in this case), the likelihood as a function of $\psi = 1/\theta$ is proportional to $\psi^3 \exp\{-\psi(x_1 + (x_2/2) + (x_3/3))\}$. The gamma($\alpha, \beta$) prior is conjugate to this likelihood form, resulting in a gamma posterior density for $\psi$ with revised parameters $\alpha + 3$ and $\beta + x_1 + (x_2/2) + (x_3/3)$.

3. Exercise 8.6.10
Solution: We have $\alpha_0/\beta_0 = \operatorname{E}(\tau) = 2$ and $\alpha_0/\beta_0^2 = \operatorname{Var}(\tau) = \operatorname{E}(\tau^2) - (\operatorname{E}(\tau))^2 = 1$, from which we can solve for $\alpha_0 = 4$ and $\beta_0 = 2$. To find $\mu_0$ and $\lambda_0$, recall that the marginal prior for $\mu$ is related to the $t_{2\alpha_0} = t_8$ distribution. In particular, from $\operatorname{E}(\mu) = 0$, we obtain $\mu_0 = 0$ (note that the mean exists and is finite, since $\alpha_0 > 1/2$). Moreover, $(2\lambda_0)^{1/2}\mu$ follows a $t_8$ distribution, and using the table of $t$ probabilities, $\Pr(|\mu| < 0.706/(2\lambda_0)^{1/2}) = 0.5$, which yields $\lambda_0 = 1/8$ from the condition given in the problem.

4. Exercise 8.6.11
Solution: Using the prior from the previous exercise, the marginal posterior distribution for $3.877(\mu - 0.988)$ is given by a $t_{18}$ distribution. Since the posterior density for $\mu$ is unimodal and symmetric, the 0.95 interval of shortest length is the equal-tail interval leaving probability 0.025 to each tail of the posterior density. Using the table of $t$ probabilities we obtain $\Pr(-2.101 < 3.877(\mu - 0.988) < 2.101 \mid x) = 0.95$, and therefore $\Pr(0.446 < \mu < 1.530 \mid x) = 0.95$.

5. Exercise 8.6.14
Solution: The answer for the Bayesian interval estimate is $\Pr(1.109 < \mu < 1.581 \mid x) = 0.9$. 
8. Consider the Rayleigh distribution as a model for the failure time (in hours) of a particular machine component. The probability density function of the Rayleigh distribution is given by

\[ f(x \mid \theta) = \theta x \exp(-0.5\theta x^2) \quad \text{for } x > 0 \]

where \( \theta > 0 \). Let \( x_i, i = 1, ..., n \), be the observed values in a random sample of failure times of the machine component, and assume an exponential prior distribution for \( \theta \), that is, \( \xi(\theta) = \beta_0 \exp(-\beta_0 \theta) \), for \( \beta_0 > 0 \).

(a) Suppose that prior information is available in the form of an estimate for the 90% percentile of the failure time distribution; in particular, it is provided that about 10% of failure times should be above 4 hours. Based on this information and using the prior predictive distribution, develop an approach to specify the prior distribution for \( \theta \), that is, to specify \( \beta_0 \).

(b) Consider an experiment where \( n = 35 \) machine components were tested producing failure times \( x_i, i = 1, ..., 35 \), such that \( \sum_{i=1}^{35} x_i^2 = 199.16 \). Derive the posterior distribution for \( \theta \) given this data set, using your prior distribution from (a). Use simulation from the posterior distribution for \( \theta \) to estimate the posterior density for the median failure time, and obtain an equal-tail interval such that the posterior probability that the median failure time lies in the interval is 0.9.

**Solution:**

(a) The prior predictive density is defined through

\[ f(x_0) = \int_0^\infty f(x_0 \mid \theta) \xi(\theta) d\theta = \beta_0 x_0 \int_0^\infty \theta \exp\{-\theta(\beta_0 + 0.5x_0^2)\} d\theta = \frac{\beta_0 x_0}{(\beta_0 + 0.5x_0^2)^{2}}, \quad \text{for } x_0 > 0 \]

from which the prior predictive distribution function can be obtained, \( F(x_0) = \int_0^{x_0} f(t) dt = 1 - (\beta_0/(\beta_0 + 0.5x_0^2)) \), for \( x_0 > 0 \). Therefore, we can match the 90% percentile of the prior predictive distribution with the prior estimate of 4 hours, \( 0.9 = F(4) = 1 - (\beta_0/(\beta_0 + 8)) \), from which we obtain \( \beta_0 = 0.889 \).

(b) The posterior density for \( \theta \) can be expressed as

\[ \xi(\theta \mid x) \propto \exp(-\beta_0 \theta) \theta^n \exp(-0.5\theta \sum_{i=1}^{n} x_i^2) \]

which can be recognized as the density of a \( \text{gamma}(n + 1, \beta_0 + 0.5\sum_{i=1}^{n} x_i^2) \) distribution. For the specific data set and prior from part (a), we have \( \xi(\theta \mid x) = \text{gamma}(36, 100.5) \).

Under the Rayleigh distribution for the failure time, the median failure time, \( \eta \), is given by

\[ 0.5 = \int_0^\eta \theta x \exp(-0.5\theta x^2)dx \]

which yields \( \eta = (2\log(2)/\theta)^{1/2} \). Therefore, if \( \{\theta_b : b = 1, ..., B\} \) is a sample from \( \xi(\theta \mid x) \) (which can be obtained using the \texttt{rgamma} R function), then \( \{\eta_b = (2\log(2)/\theta_b)^{1/2} : b = 1, ..., B\} \) is a sample from the posterior distribution for the median failure time. A histogram of the \( \{\eta_b : b = 1, ..., B\} \) provides a density estimate for \( \xi(\eta \mid x) \), and the interval estimate from the posterior sample is \( \text{Pr}(1.73 < \eta < 2.28 \mid x) = 0.9 \).