Winter 14 – AMS225 Homework 2 Solution

1. (a) The log-likelihood function can be written as

\[ \ell(\mu, \Sigma) := \log L(\mu, \Sigma \mid X_1, \ldots, X_n) \]

\[ = -\frac{n}{2} \left[ \log \det(2\pi \Sigma) + \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^T \Sigma^{-1} (X_i - \mu) \right] \]

\[ = -\frac{n}{2} \left[ \log \det(2\pi \Sigma) + \text{Tr}(\Sigma^{-1} \sum_{i=1}^{n} (X_i - \mu)(X_i - \mu)^T) \right] \]

\[ = -\frac{n}{2} \left[ \log \det(2\pi \Sigma) + \text{Tr}(\Sigma^{-1} (S_n + (\mu - \bar{X})(\mu - \bar{X})^T)) \right], \]

because

\[ \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)(X_i - \mu)^T = \frac{1}{n} \sum_{i=1}^{n} [(X_i - \bar{X}) - (\mu - \bar{X})][(X_i - \bar{X}) - (\mu - \bar{X})]^T \]

\[ = S_n + (\mu - \bar{X})(\mu - \bar{X})^T. \]

Then using the notes from class on deriving the unrestricted MLE of \((\mu, \Sigma)\),

\[ \max_{\mu \in \Omega_0, \Sigma > 0} \ell(\mu, \Sigma) = \ell(\mu_0, S_n + (\mu_0 - \bar{X})(\mu_0 - \bar{X})^T) \]

\[ = -\frac{n}{2} \left[ \log \det(2\pi (S_n + (\mu_0 - \bar{X})(\mu_0 - \bar{X})^T)) + p \right] \]

and

\[ \max_{\mu \in \Omega_1, \Sigma > 0} \ell(\mu, \Sigma) = \ell(\bar{X}, S_n) = -\frac{n}{2} \left[ \log \det(2\pi S_n) + p \right] \]

provided \(\mu_0 \neq \bar{X}\) (which occurs with probability 1 under our assumption that the \(X_i\) have a continuous distribution). So

\[ D = 2[\ell(\bar{X}, S_n) - \ell(\mu_0, S_n + (\mu_0 - \bar{X})(\mu_0 - \bar{X})^T)] \]

\[ = n \log \det(S_n + (\mu_0 - \bar{X})(\mu_0 - \bar{X})^T) - n \log \det(S_n) \]

\[ = n \log \left\{ (1 + (\mu_0 - \bar{X})^T S_n^{-1}(\mu_0 - \bar{X})) \det(S_n) \right\} - n \log \det(S_n) \]

\[ = n \log(1 + (\mu_0 - \bar{X})^T S_n^{-1}(\mu_0 - \bar{X})), \]

where the third line follows from the matrix determinant lemma.

(b) The generalized likelihood ratio test rejects \(H_0\) if

\[ D = n \log(1 + (\mu_0 - \bar{X})^T S_n^{-1}(\mu_0 - \bar{X})) \]

is large. Since \(x \to n \log(1 + x)\) is increasing in \(x\), rejecting when \(D\) is large is equivalent to rejecting when

\[ n(\mu_0 - \bar{X})^T S_n^{-1}(\mu_0 - \bar{X}) = (n - 1)(\bar{X} - \mu_0)^T S_n^{-1}(\bar{X} - \mu_0). \]

is large.
2. (a) Let

\[ B_\alpha = \{ \bar{X} + S_n^{1/2}z : z \in \mathbb{R}^p \text{ and } ||z||^2 \leq \frac{p}{n-p} \frac{F_{p,n-p}(\alpha)}{n-p} \} \]

We need to show that \( C_\alpha = B_\alpha \). First we will show that \( C_\alpha \subseteq B_\alpha \). Let \( u \in C_\alpha \). Then

\[(\bar{X} - u)^T S_n^{-1}(\bar{X} - u) \leq \frac{p}{n-p} F(\alpha) \iff ||S_n^{-1/2}(\bar{X} - u)||_2^2 \leq \frac{p}{n-p} F(\alpha),\]

Since \( u = \bar{X} + S_n^{1/2}(\bar{X} - u) \), the above implies that \( u \in B_\alpha \) with \( z = S_n^{-1/2}(\bar{X} - u) \) and hence \( C_\alpha \subseteq B_\alpha \). In the other direction, let \( v \in B_\alpha \). Then \( v = \bar{X} + S_n^{1/2}z \) for some \( z \in \mathbb{R}^p \) satisfying

\[ ||z||^2 \leq \frac{p}{n-p} F_{p,n-p}(\alpha). \]

Thus, \( v \in C_\alpha \) and hence \( B_\alpha \in C_\alpha \).

(b) The set

\[ \{ S_n^{1/2}z : ||z||^2 \leq \frac{p}{n-p} f_{p,n-p}(\alpha) \} \]

is the image of a sphere centered at 0 under the linear transformation \( z \mapsto S_n^{1/2}z \). Invoking the spectral theorem, we see that this transforms the sphere into an ellipsoid centered at 0—the sphere is stretched along axes corresponding to the eigenvectors of \( S_n \). The radii of the ellipsoid are the numbers

\[ \sqrt{\frac{p}{n-p} F_{p,n-p}(\alpha) \lambda_i} \]

where \( \lambda_i \) is the \( i \)-th eigenvalue of \( S_n \). Now by the previous part, \( C_\alpha \) is just a translate of this ellipsoid, i.e. the ellipsoid centered at \( \bar{X} \). So its radii are the same.

3. (a) We assume \( \mu = 0 \) and \( \sigma^2 = 1 \). Using the hint, let \( u(x) = g(x) \) and \( v(x) = e^{x^2/2} \). Then \( v'(x) = xe^{x^2/2} \) and integrating by parts,

\[
\text{Cov}(X, g(X)) = E[Xg(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xg(x)e^{-x^2/2}dx
\]

\[
= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x)v'(x)dx
\]

\[
= -\frac{1}{\sqrt{2\pi}} \{0 - \int_{-\infty}^{\infty} v(x)u'(x)dx\}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g'(x)e^{-x^2/2}dx
\]

\[
= Eg'(X).
\]

Note that the fourth line uses

\[
\lim_{x \to \infty} g(x)e^{-x^2/2} = 0 = \lim_{x \to \infty} g(x)e^{-x^2/2}.
\]
(b) We can reduce to the first case by considering

\[ \text{Cov}(X, g(X)) = \sigma \text{Cov}(Z, h(Z)), \]

where \( Z = \frac{(X \mu)}{\sigma} \) and \( h(z) = g(\mu + \sigma z) \). Then \( Z \sim \text{N}(0, 1) \) and

\[ \text{Cov}(Z, h(Z)) = \mathbb{E} h'(Z) = \sigma \mathbb{E} g'(X), \]

by the chain rule. Thus, \( \text{Cov}(X, g(X)) = \sigma^2 \mathbb{E} g'(X) \).