Consider the topological space $(\mathbb{R}, \sigma)$, with $\sigma$ the collection of open sets in $\mathbb{R}$ and $\mathcal{B} = \sigma(\mathcal{O})$ the Borel $\sigma$-field (in $\mathbb{R}$).

Also consider $(\overline{\mathbb{R}}, \overline{\sigma})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{ -\infty, +\infty \}$ is the two-point compactification of $\mathbb{R}$ and $\overline{\sigma}$ is the collection of open sets in $\overline{\mathbb{R}}$, consisting of the open sets in $\mathbb{R}$, sets of the form $[-\infty, x)$ for some $x \in \mathbb{R}$, sets of the form $(x, +\infty]$ for some $x \in \mathbb{R}$, and unions of sets of these types. Then $\overline{\mathcal{B}} = \sigma(\overline{\sigma})$ is the Borel $\sigma$-field (in $\overline{\mathbb{R}}$).

In order to characterize the sets in $\mathcal{B}$ through sets in $\overline{\mathcal{B}}$ we'll need the following important result:

**Lemma:** Let $\mathcal{E}$ be a class of subsets of some sample space $A$ and denote by $\mathcal{G} \cap \mathcal{A}$ the class $\{ \mathcal{B} \cap \mathcal{A} : \mathcal{B} \in \mathcal{G} \}$. Also let $\sigma(\mathcal{E})$ be the $\sigma$-field generated by $\mathcal{E}$. Then $\sigma_A(\mathcal{E} \cap \mathcal{A}) = \sigma(\mathcal{E}) \cap \mathcal{A}$ (where on the left hand side $\mathcal{A}$ rather than $\sigma$ is regarded as the entire space).
Proof

First of all we must show that \(\sigma(\mathcal{E}) \cap A = \bigcup_{G \in \sigma(\mathcal{E})} G \cap A\) is a \(\sigma\)-field of subsets of \(A\).

\(\sigma(\mathcal{E}) \cap A\) obviously forms a collection of subsets of \(A\).

- Since \(\emptyset \in \sigma(\mathcal{E})\) \(\Rightarrow \emptyset \cap A = A \in \sigma(\mathcal{E}) \cap A\) (first axiom)

- Consider \(D \in \sigma(\mathcal{E}) \cap A\) \(\Rightarrow D = G \cap A\), \(G \in \sigma(\mathcal{E})\)

  The complement of \(D\) with respect to \(A\) is

  \[
  D^c = A \cap (G \cap A) = A \cap (G \cap A)^c \quad \text{(now the complement is w.r.t. } \mathcal{E})
  \]

  \[
  = A \cap (G^c \cup A^c)
  \]

  \[
  = (A \cap G^c) \cup (A \cap A^c)
  \]

  \[
  (A \cap A^c = \emptyset)
  \]

  \[
  = G^c \cap A
  \]

  Since \(G \in \sigma(\mathcal{E})\) \(\Rightarrow G^c \in \sigma(\mathcal{E})\) \(\Rightarrow (G^c \cap A) \in \sigma(\mathcal{E}) \cap A\)

  \[
  \Rightarrow D^c \in \sigma(\mathcal{E}) \cap A
  \] (second axiom)

- Finally consider \(D_1, D_2, \ldots \in \sigma(\mathcal{E}) \cap A\)

  \[
  \Rightarrow D_i = G_i \cap A\), \(\text{with } G_i \in \sigma(\mathcal{E})\); \(i = 1, 2, \ldots\)

  Then \(\bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} (G_i \cap A)\)

  \[
  = (\bigcup_{i=1}^{\infty} G_i) \cap A \in \sigma(\mathcal{E}) \cap A
  \]

  Since \(\bigcup_{i=1}^{\infty} G_i \in \sigma(\mathcal{E})\), \(\sigma(\mathcal{E})\) being a \(\sigma\)-field (third axiom).

We have \(\mathcal{E} \subseteq \sigma(\mathcal{E})\) \(\Rightarrow \mathcal{E} \cap A \subseteq \sigma(\mathcal{E}) \cap A\) (both collections of subsets of \(A\))

\[
\Rightarrow \sigma_A(\mathcal{E} \cap A) \subseteq \sigma_A(\sigma(\mathcal{E}) \cap A) = \sigma(\mathcal{E}) \cap A
\]

since we have seen that \(\sigma(\mathcal{E}) \cap A\) is a \(\sigma\)-field of subsets of \(A\).

Thus it remains to show that \(\sigma(\mathcal{E}) \cap A \subseteq \sigma_A(\mathcal{E} \cap A)\) to establish the result. In other words we must prove that

\(G \cap A \in \sigma_A(\mathcal{E} \cap A)\), for any \(G \in \sigma(\mathcal{E})\). Following the "good sets" idea define \(\mathcal{H} = \{G \in \sigma(\mathcal{E})/ G \cap A \in \sigma_A(\mathcal{E} \cap A)\}\) \(\subseteq \sigma(\mathcal{E})\).
\( H \) is a collection of subsets of \( \mathbb{C} \). We want to show that \( H \) is a \( \sigma \)-field.

- \( A = \emptyset \land A \in \sigma_{A}(C \land A) \), since \( \sigma_{A}(C \land A) \) is a \( \sigma \)-field of subsets of \( A \), so \( \emptyset \in H \). (first axiom)

- Consider \( G \in H \Rightarrow G \land A \in \sigma_{A}(C \land A) \)
  \[ (G \land A)^{c} \in \sigma_{A}(C \land A) \] (complement w.r.t. \( A \))
  \[ \Rightarrow A \land (G \land A)^{c} \subseteq A \land (G \land A)^{c} \quad \subseteq \emptyset \]
  \[ = A \land (G^{c} \cup A^{c}) \]
  \[ = (A \land G^{c}) \cup (A \land A^{c}) \]
  \[ = G^{c} \land A \in \sigma_{A}(C \land A) \]
  \[ \Rightarrow G^{c} \in H \]. (second axiom)

- Finally consider \( G_{1}, G_{2}, \ldots \in H \Rightarrow G_{2} \land A \in \sigma_{A}(C \land A) \)
  For \( i=1,2,\ldots \)
  Then \( \bigcup_{i=1}^{\infty} (G_{i} \land A) \in \sigma_{A}(C \land A) \)
  \[ \Rightarrow (\bigcup_{i=1}^{\infty} G_{i}) \land A \in \sigma_{A}(C \land A) \Rightarrow \bigcup_{i=1}^{\infty} G_{i} \in H \]. (third axiom)

Now if \( B \in \sigma \) \( \Rightarrow B \land A \in C \land A \in \sigma_{A}(C \land A) \)
\[ \Rightarrow B \in H \]

So \( \emptyset \subseteq H \Rightarrow \sigma(\emptyset) \subseteq \sigma(H) = H \) (since \( H \) is a \( \sigma \)-field) and since from the definition \( H \subseteq \sigma(\emptyset) \), we have that \( H = \sigma(\emptyset) \), hence for any \( G \in \sigma(\emptyset) \), \( G \land A \in \sigma_{A}(C \land A) \), which completes the proof.

Now we will apply the lemma with \( \emptyset = \overline{\mathbb{R}} \), \( C = \mathbb{R} \) and \( A = \mathbb{R} \). From the form of the open sets in \( \overline{\mathbb{R}} \), we have that \( \overline{\mathbb{R}} \land \mathbb{R} = \{ \overline{\mathbb{R}} \land \overline{\mathbb{R}} \mid \overline{\mathbb{R}} \subseteq \overline{\mathbb{R}} \} = \emptyset \).

Hence \( \sigma_{\mathbb{R}}(\overline{\mathbb{R}} \land \mathbb{R}) = \sigma(\overline{\mathbb{R}}) \land \mathbb{R} \Rightarrow \sigma_{\mathbb{R}}(\emptyset) = \overline{\mathbb{R}} \land \mathbb{R} \)
Thus a Borel set $B$ in $\mathbb{R}$ is equal to the intersection of a Borel set $\overline{B}$ in $\mathbb{R}^c$ and $\mathbb{R}$.

Now since $\overline{B} \subseteq \mathbb{R} = \mathbb{R} \cup (-\infty, \infty)$, we have

\[
\overline{B} = (\overline{B} \cap \mathbb{R}) \cup (\overline{B} \cap (-\infty, \infty)) = (\overline{B} \cap R^c) \cup B
\]

(comelement w.r.t. $\mathbb{R}$)

The intersection $\overline{B} \cap (-\infty, \infty)$ can be:

- $\emptyset$ if $B \subseteq R^c$,
- $(-\infty, \infty)$ if, for example, $B = (-\infty, x)$,
- for some $x \in \mathbb{R}$,
- or $\{x\}$ if, for example, $B = (x, \infty]$, for some $x \in \mathbb{R}$,
- or finally $(-\infty, 0)$ if, for example, $B = (-\infty, x) \cup (y, \infty)$, for $x, y \in \mathbb{R}$.

Hence, in conclusion, a Borel subset of $\mathbb{R}$ is characterized as the union of a Borel subset of $\mathbb{R}$ and one of the four subsets of $(-\infty, 0)$.

Borel subsets of $\mathbb{R}^+$

Topological space $(\mathbb{R}^+, \mathcal{O}^+)$, where $\mathbb{R}^+ = \mathbb{R}_+ \cup \{\infty\}$ and $\mathcal{O}^+$ the collection of open sets in $\mathbb{R}^+$ consisting of the open sets in $\mathbb{R}_+$ (i.e., sets of the form $(a, b)$, $a > 0$, $[0, b)$ and unions of such), sets of the form $[a, \infty)$, for some $a > 0$ and unions of sets of these types.

The Borel $\sigma$-field is $\mathcal{B}^+ = \sigma(\mathcal{O}^+)$

Now apply the lemma with $\mathcal{O} = \mathbb{R}^+$, $\mathcal{C} = \mathcal{O}^+$ and $A = \mathbb{R}^+$

We have $\mathcal{O}^+ \cap \mathbb{R}^+ = \{\mathcal{O}^+ \cap \mathbb{R}^+ : \mathcal{O}^+ \in \mathcal{O}^+\} = \mathcal{O}^+$ the collection of open sets in $\mathbb{R}^+$. Then $\sigma_{\mathbb{R}^+}(\mathcal{O}^+ \cap \mathbb{R}^+) = \sigma(\mathcal{O}^+) \cap \mathbb{R}^+$.
\[ \sigma_{\mathbb{R}^+}(0^+) = \overline{B}^+ \cap \mathbb{R}^+ \Rightarrow \mathbb{B}^+ = \overline{B}^+ \cap \mathbb{R}^+ \]

with \( \mathbb{B}^+ \) the Borel \( \sigma \)-field of \( \mathbb{R}^+ \) which can be described as the collection of the Borel subsets of \( \mathbb{R} \) that contain only nonnegative numbers. To see this we can apply again the lemma with \( \mathfrak{O} = \mathbb{R} \), \( \mathcal{G} = \emptyset \), and \( A = \mathbb{R}^+ \) (now \( \emptyset \cap \mathbb{R}^+ = 0^+ \)) to get
\[ \mathbb{B}^+ = \mathbb{B} \cap \mathbb{R}^+ = \sum \{ B \cap \mathbb{R}^+ : B \in \mathbb{B} \} \]

Returning to the relation \( \mathbb{B}^+ = \overline{B}^+ \cap \mathbb{R}^+ \), we can write a Borel subset \( \overline{B}^+ \) of \( \mathbb{R}^+ \) in the form
\[
\overline{B}^+ = (\overline{B}^+ \cap \mathbb{R}^+) \cup (\overline{B}^+ \cap \mathbb{R}^+) \cup (\overline{B}^+ \cap \mathbb{R}^+) \cup B^+ \]

\[ = (\overline{B}^+ \cap \overline{\{0\}}) \cup B^+ \]

\[ = \begin{cases} \emptyset \cup B^+ = B^+ \\ B^+ \cup \mathbb{\infty} \cap \overline{\{0\}} \end{cases} \]

Hence a Borel subset of \( \overline{\mathbb{R}^+} \) is either equal to a Borel subset of \( \mathbb{R}^+ \) or to the union of a Borel subset of \( \mathbb{R}^+ \) and \( \mathbb{\infty} \).