AMS 261: Probability Theory (Winter 2014)

Homework 1 (due Thursday 1/23)

1. Consider a sample space $\Omega$.
   (a) Prove that any intersection of $\sigma$-fields (of subsets of $\Omega$) is a $\sigma$-field. That is, if $F_j, j \in J$, are $\sigma$-fields on $\Omega$ (with $J$ an arbitrary index set, countable or uncountable), then show that $F = \bigcap_{j \in J} F_j$ is a $\sigma$-field.
   (b) Show by counterexample that a union of $\sigma$-fields may not be a $\sigma$-field.

2. Given a sample space $\Omega$ and a collection $E$ of subsets of $\Omega$, the $\sigma$-field generated by $E$, $\sigma(E)$, is defined as the intersection of all $\sigma$-fields on $\Omega$ that contain $E$. (As shown in class, $\sigma(E)$ is the smallest $\sigma$-field that contains $E$.)
   (a) Consider two collections $E_1$ and $E_2$ of subsets of $\Omega$. Show that if $E_1 \subseteq E_2$, then $\sigma(E_1) \subseteq \sigma(E_2)$.
   (b) As in part (a), let $E_1$ and $E_2$ be collections of subsets of the sample space $\Omega$. Prove that if $E_1 \subseteq \sigma(E_2)$ and $E_2 \subseteq \sigma(E_1)$, then $\sigma(E_1) = \sigma(E_2)$.

3. Let $\mathcal{F}$ be a collection of subsets of a sample space $\Omega$.
   (a) Suppose that $\Omega \in \mathcal{F}$, and that when $A, B \in \mathcal{F}$ then $A \cap B^c \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.
   (b) Suppose that $\Omega \in \mathcal{F}$, and that $\mathcal{F}$ is closed under the formation of complements and finite pairwise disjoint unions. Show by counterexample that $\mathcal{F}$ need not be a field.

4. Consider the sample space $\Omega = (0, 1]$ and the collection $B_0$ of all finite pairwise disjoint unions of subintervals of $(0, 1]$. That is, any member $B$ of $B_0$ is of the form $B = \bigcup_{i=1}^{n} (a_i, b_i]$, where for each $i = 1, ..., n$, $0 \leq a_i < b_i \leq 1$, and $(a_i, b_i] \cap (a_j, b_j] = \emptyset$ for any $i \neq j$.
   Show that $B_0$ augmented by the empty set is a field, but not a $\sigma$-field.

5. Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be a countable set, $\{p_n : n = 1, 2, \ldots\}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} p_n = 1$, and $\mathcal{F}$ be the collection of all subsets of $\Omega$. For each $A \in \mathcal{F}$, define the set function
   \[
P(A) = \sum_{\{n : \omega_n \in A\}} p_n.
   \]
   Show that $(\Omega, \mathcal{F}, P)$ is a probability space.