1. Consider a sequence \( \{X_n : n = 1, 2, ...\} \) of \( \mathbb{R} \)-valued random variables defined on the same probability space \((\Omega, \mathcal{F}, P)\). Assume that the sequence is (pointwise) increasing, that is, for all \( n \) and for each \( \omega \in \Omega \), \( X_n(\omega) \leq X_{n+1}(\omega) \). Moreover, assume that \( E(X_1) > -\infty \). Denote by \( X \) the pointwise limit of \( \{X_n : n = 1, 2, ...\} \), that is, for each \( \omega \in \Omega \), \( X(\omega) = \lim_{n \to \infty} X_n(\omega) \).

- Prove that \( E(X) = \lim_{n \to \infty} E(X_n) \).

2. Let \( \{X_n : n = 1, 2, ...\} \) be a countable sequence of \( \mathbb{R}^+ \)-valued random variables defined on a common probability space \((\Omega, \mathcal{F}, P)\), and assume that \( E(\sum_{n=1}^{\infty} X_n) < \infty \).

- Show that \( E \left( \sum_{n=1}^{\infty} X_n \right) = \sum_{n=1}^{\infty} E(X_n) \).

3. Let \( \{X_n : n = 1, 2, ...\} \), \( \{Y_n : n = 1, 2, ...\} \), and \( \{Z_n : n = 1, 2, ...\} \) be sequences of \( \mathbb{R} \)-valued random variables (all the random variables are defined on the same probability space). Assume that:
  (a) \( E(X_n) \) and \( E(Z_n) \) exist for all \( n \) and are finite;
  (b) each of the three sequences converges almost surely (denote by \( X, Y, \) and \( Z \) the respective almost sure limits);
  (c) \( E(X), E(Y), \) and \( E(Z) \) exist and are finite;
  (d) \( X_n \leq Y_n \leq Z_n \) almost surely;
  (e) \( \lim_{n \to \infty} E(X_n) = E(X) \), and \( \lim_{n \to \infty} E(Z_n) = E(Z) \).

- Show that \( \lim_{n \to \infty} E(Y_n) = E(Y) \).

4. Let \( \{X_n : n = 1, 2, ...\} \) be a countable sequence of \( \mathbb{R} \)-valued random variables defined on a common probability space \((\Omega, \mathcal{F}, P)\). Assume that there exist finite real constants \( p > 1 \) and \( K > 0 \) such that \( \sup_n E(|X_n|^p) \leq K \).

- Show that \( \{X_n : n = 1, 2, ...\} \) is uniformly integrable.

5. Let \( X \) be an \( \mathbb{R} \)-valued random variable, defined on a probability space \((\Omega, \mathcal{F}, P)\), with finite expectation \( \mu = E(X) \) and finite standard deviation \( \sigma = (\text{Var}(X))^{1/2} \).

- Prove that for any \( 0 \leq z \leq \sigma \),

\[
P \left( \{ \omega \in \Omega : |X(\omega) - \mu| \geq z \} \right) \geq \frac{\sigma^4 \{ 1 - (z/\sigma)^2 \}^2}{E(|X - \mu|^4)}.
\]

6. Let \( \{X_n : n = 1, 2, ...\} \) be a sequence of \( \mathbb{R} \)-valued random variables defined on a common probability space \((\Omega, \mathcal{F}, P)\). Suppose there exists an \( \mathbb{R}^+ \)-valued random variable \( Y \), defined on \((\Omega, \mathcal{F}, P)\), such that \( E(Y) < \infty \) and \( |X_n| \leq Y \), almost surely, for all \( n \).

- Show that \( \{X_n : n = 1, 2, ...\} \) is uniformly integrable.

7. Consider a countable sequence \( \{X_n : n = 1, 2, ...\} \) of \( \mathbb{R} \)-valued random variables, defined on a common probability space \((\Omega, \mathcal{F}, P)\), and an increasing function \( G : [0, \infty) \to [0, \infty) \), which satisfies \( \lim_{t \to \infty} \{ t^{-1} G(t) \} = \infty \) and \( 0 < \sup_n E\{G(|X_n|)\} < \infty \).

- Prove that \( \{X_n : n = 1, 2, ...\} \) is uniformly integrable.