1. Consider real-valued random variables \( A_i, B_i \), \( i = 1, \ldots, k \), such that \( \text{E}(A_i) = \text{E}(B_i) = 0 \) and \( \text{Var}(A_i) = \text{Var}(B_i) = \sigma_i^2 > 0 \), for \( i = 1, \ldots, k \). Moreover, assume they are mutually uncorrelated, that is, \( \text{E}(A_i A_l) = \text{E}(B_i B_l) = 0 \), for \( i \neq l \), and \( \text{E}(A_i B_l) = 0 \), for all \( i, l \). Define the stochastic process \( X = \{X_t : t \in \mathbb{R}\} \) by \( X_t = \sum_{i=1}^k (A_i \cos(w_i t) + B_i \sin(w_i t)), \) where \( w_i, i = 1, \ldots, k \), are real constants. Show that \( X \) is weakly stationary but not strongly stationary.

2. Consider a discrete-time real-valued stochastic process \( X = \{X_n : n \geq 1\} \) defined by \( X_n = \cos(n U), \) where \( U \) is uniformly distributed on \( (-\pi, \pi) \). Show that \( X \) is weakly stationary but not strongly stationary.

3. Let \( \{Z_n\} \), for integer \( n \), be a sequence of real-valued random variables with \( \text{E}(Z_n) = 0 \), \( \text{Var}(Z_n) = 1 \) and \( \text{E}(Z_n Z_m) = 0 \), \( n \neq m \). Consider a moving average process associated with \( \{Z_n\} \), that is, a discrete-time real-valued process \( Y = \{Y_n\} \), with integer \( n \), given by \( Y_n = Z_n + a Z_{n-1} \), where \( a \) is a real constant. Show that \( Y \) is weakly stationary and find its covariance function. Obtain the spectral density function of \( Y \).

4. Consider a weakly stationary process \( X = \{X_t : t \in \mathbb{R}\} \) with zero mean and unit variance. Find the correlation function of \( X \) if the spectral density function \( f \) of \( X \) is given by:
   (a) \( f(u) = (2\pi)^{-1/2} \exp(-0.5u^2), \) \( u \in \mathbb{R} \).
   (b) \( f(u) = 0.5 \exp(-|u|), \) \( u \in \mathbb{R} \).

5. Show that a Gaussian process is strongly stationary if and only if it is weakly stationary.

6. Let \( W = \{W_t : t \geq 0\} \) be a continuous-time, real-valued Gaussian process such that:
   (a) \( W_0 = \omega \), where \( \omega \) is a real constant.
   (b) \( W \) has independent increments, that is, the random variables \( W_{t_1} - W_{s_1}, \ldots, W_{t_n} - W_{s_n} \) are independent whenever the intervals \( (s_j, t_j], \) \( j = 1, \ldots, n \), are disjoint.
   (c) \( W_{s+t} - W_s \) follows a \( N(0, \sigma^2 t) \) distribution, for all \( s, t \geq 0 \), where \( \sigma^2 \) is a positive constant. Define the continuous-time, real-valued stochastic process \( X = \{X_t : t \geq 1\} \), by \( X_t = W_t - W_{t-1} \). Show that \( X \) is strongly stationary and find its spectral density function.

7. **Gaussian Markov processes.** By definition, a continuous-time real-valued stochastic process \( X = \{X_t : t \in \mathbb{R}\} \) is called a Markov process if for all \( n \), for all \( x, x_1, \ldots, x_{n-1} \), and all increasing sequences \( t_1 < t_2 < \ldots < t_n \) of index points,
   \[
   \Pr(X_{t_n} \leq x \mid X_{t_1} = x_1, \ldots, X_{t_{n-1}} = x_{n-1}) = \Pr(X_{t_n} \leq x \mid X_{t_{n-1}} = x_{n-1}).
   \]
   Consider a (real-valued) Gaussian process \( Y = \{Y_t : t \in \mathbb{R}\} \). Show that \( Y \) is a Markov process if and only if \( \text{E}(Y_{t_n} \mid Y_{t_1} = y_1, \ldots, Y_{t_{n-1}} = y_{n-1}) = \text{E}(Y_{t_n} \mid Y_{t_{n-1}} = y_{n-1}) \), for all \( n, y_1, \ldots, y_{n-1} \) and all increasing sequences \( t_1 < t_2 < \ldots < t_n \) of index points.

8. **Stationary Gaussian Markov processes.** Consider a continuous-time real-valued stochastic process \( X = \{X_t : t \geq 0\} \), which is assumed to be Gaussian, stationary (with mean 0), and Markov. Show that the covariance function \( c(\cdot) \) of \( X \) satisfies the functional equation
   \[
c(0)c(t_1 + t_2) = c(t_1)c(t_2), \quad \forall t_1, t_2 \geq 0.
   \]