Unless explicitly stated otherwise, assume that a “Markov chain” refers to a discrete-time \((T = \{0, 1, 2, \ldots\})\), discrete-state \((S \subseteq \mathbb{Z})\) Markov chain, \(X = \{X_n : n \geq 0\}\), such that

\[
\Pr(X_{n+1} = s \mid X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \Pr(X_{n+1} = s \mid X_n = x_n)
\]

for all \(n \geq 1\) and all \(s, x_0, x_1, \ldots, x_n \in S\). Moreover, for problems 2 – 7, assume also time-homogeneity for transition probabilities.

1. Show that the Markov property (1) is equivalent to each of the following conditions:
   (a) For all \(n, m \geq 1\) and all \(s, x_0, x_1, \ldots, x_n \in S\),
   \[
   \Pr(X_{n+m} = s \mid X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \Pr(X_{n+m} = s \mid X_n = x_n)
   \]
   (b) For all \(0 \leq n_1 < n_2 < \ldots < n_k \leq n\), all \(m \geq 1\) and all \(s, x_1, \ldots, x_k \in S\),
   \[
   \Pr(X_{n+m} = s \mid X_{n_1} = x_1, X_{n_2} = x_2, \ldots, X_{n_k} = x_k) = \Pr(X_{n+m} = s \mid X_{n_k} = x_k)
   \]
   (c) For all \(n > 1\), \(x_n \in S\), all finite sets of time points \(\{\ell \in L : 0 \leq \ell < n\}\) and \(\{k \in K : n < k\}\), and all corresponding sets of states \(\{x_k : k \in K\}\), \(\{x_\ell : \ell \in L\}\),
   \[
   \Pr(\{X_k = x_k : k \in K\}, \{X_\ell = x_\ell : \ell \in L\} \mid X_n = x_n) = \Pr(\{X_k = x_k : k \in K\} \mid X_n = x_n)\Pr(\{X_\ell = x_\ell : \ell \in L\} \mid X_n = x_n)
   \]
   (that is, “given the present, the future is independent of the past”).

2. Assume \(X = \{X_n : n \geq 0\}\) is a Markov chain and let \(\{n_k : k \geq 0\}\) be an unbounded increasing sequence of positive integers. Define a new stochastic process \(Y = \{Y_k : k \geq 0\}\) such that \(Y_k = X_{n_k}\). Show that \(Y\) is a Markov chain. Is \(Y\) a time-homogeneous Markov chain without additional conditions?

3. Consider a Markov chain with state space \(S = \{1, 2, 3, 4\}\) and transition matrix
   \[
   \begin{pmatrix}
   0 & 1 & 0 & 0 \\
   \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
   \frac{1}{3} & 0 & 0 & \frac{2}{3} \\
   1 & 0 & 0 & 0 \\
   0 & 0 & 1 & 0
   \end{pmatrix}
   \]
   Classify the states of the chain, that is, for each state, determine whether it is transient or persistent (and if so, whether it is null or positive persistent), and also whether it is aperiodic or periodic (and if so, find the period). Obtain the mean recurrence time for each state.
4. Consider a Markov chain with state space \( S = \{1, 2, 3, 4\} \) and transition matrix

\[
\begin{pmatrix}
\frac{1}{3} & \frac{2}{3} & 0 & 0 \\
\frac{3}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Classify the states of the chain. Compute \( f_{34} \), that is, the probability of ultimate absorption in state 4, starting from state 3.

5. Consider a Markov chain with three states \( \{1, 2, 3\} \) and transition matrix

\[
\begin{pmatrix}
1 - 2p & 2p & 0 \\
p & 1 - 2p & p \\
0 & 2p & 1 - 2p
\end{pmatrix}
\]

where \( 0 < p < 0.5 \). Classify the states of the chain. Obtain \( p_{ii}^{(n)} \) for \( i = 1, 2, 3 \). Calculate the mean recurrence times \( \mu_i, i = 1, 2, 3 \), of the states.

6. Recall that the transition matrix \( P \) of a Markov chain is stochastic, that is, all its entries \( p_{ij} \) are non-negative and \( \sum_{j \in S} p_{ij} = 1 \), for all \( i \in S \). The transition matrix is doubly stochastic if it also satisfies \( \sum_{i \in S} p_{ij} = 1 \), for all \( j \in S \).

Consider an irreducible Markov chain \( X \) with finite state space \( S \) of dimension \( K = |S| < \infty \), and transition matrix \( P \). Show that \( X \) has a discrete uniform stationary distribution (that is, \( \pi_i = K^{-1} \) for each \( i \in S \)) if and only if \( P \) is doubly stochastic.

7. **Strong Markov property.** Assume \( X = \{X_n : n \geq 0\} \) is a Markov chain. Let \( W \) be a random variable (random time), taking values in \( \{0, 1, 2, ...\} \), with the property that the indicator function of the event \( \{W = n\} \) is a function of the variables \( X_0, X_1, ..., X_n \) only. Such a random variable \( W \) is called a stopping time. The definition requires that we can determine probabilities for its values, \( W = n \), with knowledge only of the past and the present, \( X_0, X_1, ..., X_n \), and with no further information about the future.

Prove that

\[
\Pr(X_{W+m} = j \mid X_k = x_k, 0 \leq k < W, X_W = i) = \Pr(X_{W+m} = j \mid X_W = i),
\]

for all \( m \geq 1 \) and all \( x_k, i, j \in S \).

**Note:** It will help with the proof if you obtain first the following lemma.

*Lemma:* Consider a Markov chain \( X = \{X_n : n \geq 0\} \) and let \( I : S^m \to \{0, 1\} \) be a function that assigns the value of 0 or 1 to each collection of states \( (x_1, ..., x_n) \). Then, for any \( m \geq 1 \), the distribution of \( X_{n+m} \) conditional on \( \{I(X_1, ..., X_n) = 1\} \cap \{X_n = i\} \) is equivalent to the distribution of \( X_{n+m} \) conditional on \( \{X_n = i\} \).