Outline

The addition rule
  More on Independence
  Components in Series
  Components in Parallel
  Summary of Probability Rules

The Binomial Formula
  Typical Problems
The addition rule

We saw that the multiplicative rule is useful when looking at two events that occur jointly. So it is related to the problem of observing $A$ and $B$. Let’s consider the or case.

Mutually exclusive events

Two events are mutually exclusive or disjoint when the occurrence of one prevents the occurrence of the other.
Two dice are rolled and the sum of the two is observed. The events: the sum is greater than 6 and the sum is smaller than 3 are disjoint.

Addition Rule and Disjoint events

If two events are disjoint then, the probability that at least one will happen is obtained by adding the probabilities of each event.
Q: What are the chances that a card from a well shuffled deck will be either hearts or spades?

A: We either think that there are 26 out of the 52 cards that are either spades or hearts, and so the probability is $26/52 = 1/2$.

Or we can think that the chance of the card being hearts is $1/4$ and the chance of it being spades is $1/4$. Since the two events are disjoint, the chance of either spades or hearts is $1/4 + 1/4 = 1/2$.

Q: What are the chances of getting at least one 1 when two dice are rolled?

A: Consider the events: 1 in the red die and one in the blue die. If we use the addition rule then the chances are $1/6 + 1/6 = 1/3 = 12/36$. But if we observe the table we have that there are 11 ways of getting a 1. Thus the chances are just $11/36$. 
The addition rule does not apply in this case since the two events are not disjoint. The addition rule is counting the outcome 1 1 twice.

Note:

If two events are not disjoint and the addition rule is used the result will be too big.

The right answer is obtained by subtracting the probability that the events happen simultaneously.

In the dice example we have $1/6 + 1/6 - 1/36 = 11/36$, which is the correct answer.
The mathematical notation for this is

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]
More on Independence

Q: If two events are disjoint, are they also independent?

A: No. Actually when two events are disjoint, the occurrence of one gives a lot of information about the occurrence of the other. In particular, the other can not happen!

Remember that two events are independent when the probability of one is unaffected by the other event happening. Since this is not the case when events are disjoint, then they are not independent.
Suppose $A$ is an event and denote $A^c$ as the opposite of $A$. Then $A$ and $A^c$ are mutually exclusive but, for any value of $P(A)$, $P(A|A^c) = 0$, so they are not independent.

Independent $\iff$ multiply unconditional probabilities

Disjoint $\iff$ add probabilities
What is wrong with this reasoning?

**Q:** What is the probability that in four rolls of a die at least one 1 will turn up?

**A:** In one roll of a die there is $1/6$ probability of getting a 1. In 4 rolls there is $4 \times 1/6 = 2/3$.

**Q:** What is the probability that in 24 rolls of a pair of dice, at least one double 1 will turn up?

**A:** In one roll of a pair of dice there is $1/36$ chance of getting a double 1. In 24 rolls the chances are $24 \times 1/36 = 2/3$.

We are adding probabilities for non mutually exclusive events.

To see how wrong our argument is, think of rolling one die 8 times, then we would get $8 \times 1/6 > 1$ probability of observing a 1!
What is the right calculation?

**First problem:** Think of the opposite event. The gambler looses if none of the four rolls come up 1. What are the chances of not getting a 1 in a specific roll? This can be calculated as \(1 - \frac{1}{6} = \frac{5}{6}\). For the gambler to lose this has to happen the first \textbf{and} the second \textbf{and} the third \textbf{and} the fourth. The rolls are all independent, thus we use the multiplication rule and obtain \(\left(\frac{5}{6}\right)^4 = 0.482\). Thus the chance of winning is \(1 - 0.482 = 0.518\).

**Second problem:** In a similar way we get the chances of the event which is the opposite of getting a double 1 in one particular roll: \(\frac{35}{36}\). We want this to happen 24 times, so we get \(\left(\frac{35}{36}\right)^{24} = 0.509\). The chance of winning is \(1 - 0.509 = 0.491\).
Components in Series

A system of three components in series is such that the whole system works if all the components work. That corresponds to the figure
If all three components are independent and they have probability $p_1, p_2$ and $p_3$ of working properly, then the system will work properly if all three components work and that, using the multiplication rule, happens with probability

$$P = p_1p_2p_3$$

This can be generalized to an arbitrary number of components.
Components in Parallel

Suppose that a system is made of three component and it is such that it will work if any of the three components work properly. Then the components are in parallel.
To work out the probability that the system will function, consider the opposite event: the system will not function. This only happens if all three components do not function.
Summary of Probability Rules

For any event $A$

$$0 \leq P(A) \leq 1$$

The probability for the “opposite” of $A$

$$P(\text{not } A) = 1 - P(A)$$

The addition rule for any two events $A$ and $B$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If $A$ and $B$ are mutually exclusive, then $P(A \text{ and } B) = 0$ and the addition rule simplifies to

$$P(A \text{ or } B) = P(A) + P(B)$$
Probability

The multiplication rule for any two events $A$ and $B$

$$P(A \text{ and } B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$$

when $A$ and $B$ are independent then

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

so that the multiplication rule becomes

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Notice that from the multiplication rule we obtain a definition of conditional probability given by

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$
The Binomial Formula

Typical Problems

Suppose that we can attempt a given number of trials or experiments, as for example, roll a die 10 times. In this class we will study how to calculate the chances of getting a given outcome (e.g., two heads) from the total number of attempts or trials.
1. A coin is tossed six times. What is the chance of getting two heads?

2. A die is rolled eight times. What is the chance of getting three times one dot?

3. A box contains one red marble and nine green ones. Five draws are made at random with replacement. What is the chance of getting two red marbles in five draws?
We will continue with the example of the red marbles. One possibility of getting two red marbles in five draws is to get the sequence:

$$RRGGG$$

In this case the first two draws are red and the last three green. Another possibility is to get the second and the fifth draw reds as follows:

$$GRGGR$$
To calculate the chance of the first pattern we apply the multiplication rule as follows:

\[
P(RRGGG) = \frac{1}{10} \times \frac{1}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \\
= \left(\frac{1}{10}\right)^2 \times \left(\frac{9}{10}\right)^3
\]
To calculate the chance of the pattern $GRGGR$ we apply the multiplication rule in a similar way as follows:

$$P(GRGGR) = \frac{9}{10} \times \frac{1}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{1}{10}$$

$$= \left(\frac{1}{10}\right)^2 \times \left(\frac{9}{10}\right)^3$$

The pattern $RRGGG$ has the same chance as the pattern $GRGGR$. The same is true for all possible patterns with two R’s and three G’s.
The Binomial Coefficient

Number of Patterns

The number of patterns is given by the binomial coefficient

The binomial coefficient is calculated as:

\[
\frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (3 \times 2 \times 1)} = 10
\]

In mathematical form this can be written as:

\[
\frac{5!}{2!3!}
\]
The exclamation mark is read *factorial*. For example, the number of ways to arrange four R’s and one G in a row is:

\[
\frac{5!}{4!1!} = 5
\]

The number of ways to arrange five R’s and zero G’s is:

\[
\frac{5!}{5!0!} = 1
\]

The convention is that 0! equals 1. Note that there is only one way of getting five R’s and zero G’s:

\[RRRRR\]
Another Formula

The sum of the chances of all patterns equals the number of patterns times the common chance
A die is rolled four times. Find the chance that one dot appears exactly twice.

Using the binomial coefficient we get the number of ways to get one dot exactly twice:

\[
\frac{4!}{2!2!} = 6
\]

The chance that one dot appears exactly twice is:

\[
6 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 = \frac{150}{1,296} \sim 12\%
\]
The results presented previously can be summarized in the *binomial formula*.

Suppose that $n$ is the number of trials, as for example, rolling a die ten times. Let $k$ be equal to the number of times a given event is to occur, and $p$ is the probability that the event will occur on any particular trial. The *binomial formula* can be written as

$$
\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}
$$
Assumptions in the Application

1. The value of $n$ must be fixed in advance

2. $p$ must be equal from trial to trial

3. The trials are independent
Of families with 4 children, what proportion have more girls than boys? You may assume that the sex of a child is determined as if by drawing at random with replacement from:

\[
\begin{array}{c|c}
M & F \\
\hline
\end{array}
\]

where M=male and F=female.

To calculate the proportion or chances to get more girls than boys, we need to calculate the chance of three girls and four girls in families with four children.
The chance of having three girls with the above assumptions is:

\[
\frac{4!}{3!1!} \left( \frac{1}{2} \right)^3 \left( \frac{1}{2} \right)^1 = 25\% 
\]

The chance of having four girls is:

\[
\frac{4!}{4!0!} \left( \frac{1}{2} \right)^4 \left( \frac{1}{2} \right)^0 \sim 6\% 
\]

The proportion of families with more girls than boys is the sum of the above percentages: \( \sim 31\% \).