AMS 7
Analysis of Variance
Lecture 17

Department of Applied Mathematics and Statistics, University of California, Santa Cruz

Fall 2013
Two-sample t-tests are a way of seeing if two groups have the same population mean.

Test statistic: \[ t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \]

Can we tell from the sample data whether the two population means are different? i.e. is the difference in sample means (between the sample groups) large relative to the variability within sample groups?

What if we have more than two groups to compare?

ANOVA (ANalysis Of VAriance) generalizes this for \( k \) sample groups. It tests for equality of population means for more than two populations.
• How could we compare three brands of cookies?
  → We don’t want to do lots of pairwise tests as the probability of a type I error for the combined test will become larger.

<table>
<thead>
<tr>
<th></th>
<th>size</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( n_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean ( \bar{x} )</td>
<td>( \bar{x}_1 )</td>
<td>( \bar{x}_2 )</td>
<td>( \bar{x}_3 )</td>
<td></td>
</tr>
<tr>
<td>std. dev. ( s )</td>
<td>( s_1 )</td>
<td>( s_2 )</td>
<td>( s_3 )</td>
<td></td>
</tr>
</tbody>
</table>

★ The overall mean is
\[
\bar{X} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2 + n_3 \bar{x}_3}{n_1 + n_2 + n_3} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2 + n_3 \bar{x}_3}{N} = \frac{\sum_{i=1}^{k} n_i \bar{x}_i}{\sum_{i=1}^{k} n_i}
\]

★ The total sum of squares - \( SS(\text{total}) \) - is the overall variability in the data
\[
SS(\text{total}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2
\]

★ The total can be broken down into two parts:
\[
SS(\text{total}) = SS(\text{treatment}) + SS(\text{error})
\]
• SS(treatment) is the sum of squares due to treatment, or the sum of squares between sample groups.

\[ SS(\text{treatment}) = \sum_{i=1}^{k} n_i (\bar{x}_i - \bar{x})^2 \]

→ It measures variability between the groups (sample means).

• SS(error) is the sum of squares due to error, or the sum of squares within sample groups.

\[ SS(\text{error}) = \sum_{i=1}^{k} (n_i - 1)s_i^2, \]

where \( s_i^2 = \frac{\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2}{n_1 - 1} \)

→ It measures variability within groups - natural error.
We use these to do the hypothesis test.

\[ H_0 : \mu_1 = \mu_2 = \ldots = \mu_k \]

where \( \mu_i \) is the population mean for group \( i \in \{1, \ldots, k\} \)

1. \( H_1 : \) not all population means equal (some could be equal, but not all)

2. \( \alpha = 0.05 \)

3. test statistic: \( F = \frac{MS(\text{treatment})}{MS(\text{error})} = \frac{SS(\text{treatment})/k-1}{SS(\text{error})/N-k} \), sampling distribution is \( F \) with \( k-1, N-k \) df.

4. Get critical values for Table A-5, or get p-value from JMP

5. Reject or fail to reject \( H_0 \)

6. Draw conclusions

Typically make an ANOVA table:

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>SS</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>SS(treatment)</td>
<td>( k-1 )</td>
<td>( MS(tr) = \frac{SS(tr)}{k-1} )</td>
<td>( MS(tr) )</td>
</tr>
<tr>
<td>Error</td>
<td>SS(error)</td>
<td>( N-k )</td>
<td>( MS(err) = \frac{SS(err)}{N-k} )</td>
<td>-</td>
</tr>
<tr>
<td>Total</td>
<td>SS(total)</td>
<td>( N-1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example: Students at a certain community college have a pre-algebra course where they take a math competency exam at the end of the course. There are three types of course a student can enroll in: 4 days/week for 17 weeks, 5 days/week for 17 weeks, or 5 days/week for 4 intensive weeks.

The following exam data was collected:

<table>
<thead>
<tr>
<th></th>
<th>4D/w</th>
<th>5D/w</th>
<th>Intensive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample size</td>
<td>355</td>
<td>93</td>
<td>115</td>
</tr>
<tr>
<td>mean</td>
<td>41.1</td>
<td>37.5</td>
<td>44.4</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>4.94</td>
<td>5.73</td>
<td>5.13</td>
</tr>
</tbody>
</table>

\[ N = 563 \]

Test the claim that the three courses produce the same mean exam scores.
Test the claim that the three courses produce the same mean exam scores.

\( H_0 : \mu_4 = \mu_5 = \mu_I \)

where \( \mu_i \) is the population mean exam scores for group \( i \in \{4, 5, I\} \)

1. 

\( H_1 : \) not all \( \mu \)'s equal

2. \( \alpha = 0.05 \)

3. test statistic: 

\[
F = \frac{MS(\text{treatment})}{MS(\text{error})}, \quad F \text{ with } k - 1 = 2, \quad N - k = 560 \text{ df.}
\]

4. From Table A-5 critical value is \( F = 3.00 \), reject \( H_0 \) if \( F > 3.00 \)

ANOVA table (from JMP):

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>2454</td>
<td>2</td>
<td>1227</td>
<td>48.5</td>
</tr>
<tr>
<td>Error</td>
<td>14155</td>
<td>560</td>
<td>25.3</td>
<td>-</td>
</tr>
<tr>
<td>Total</td>
<td>16609</td>
<td>562</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. \( 48.5 < 3.00 \), so reject \( H_0 \)

6. Conclude that not all courses produce the same mean exam scores. (requires more work to say which ones are different)
Key Concepts!!!!!

- ANOVA