AMS7: WEEK 2. CLASS 2


Friday April 10, 2015
Probability: Introduction

- Probability: Underlying foundation of inferential statistics

- DEFINITIONS:
  1) **AN EVENT**: Any collection of results or outcomes of a procedure. EXAMPLE: Tossing a die (a procedure) and getting even numbers: A={2,4,6}
  2) **A SIMPLE EVENT**: It is an outcome or event that cannot be further broken down into simple pieces. EXAMPLE: Outcomes when you roll a die: {1} or {2} or {3} or {4} or {5} or {6}
  3) **SAMPLE SPACE**: All possible simple events for a procedure. EXAMPLE: Tossing a die. Possible outcomes are S={1,2,3,4,5,6}
NOTATION

- P denotes Probability
- A, B, C denote specific events
- P(A) denotes the probability of event A occurring

EXAMPLE:
- Procedure: Rolling two dice.
- SIMPLE EVENTS:
  \{1,1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,1\}, \{2,2\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,1\}, \{3,2\}, \{3,3\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,1\}, \{4,2\}, \{4,3\}, \{4,4\}, \{4,5\}, \{4,6\}, \{5,1\}, \{5,2\}, \{5,3\}, \{5,4\}, \{5,5\}, \{5,6\}, \{6,1\}, \{6,2\}, \{6,3\}, \{6,4\}, \{6,5\}, \{6,6\}

36 possibilities! (6×6)
EXAMPLE: Rolling two dice

- SAMPLE SPACE: Consists of all possible simple events
- Particular Event A: Getting doubles
- Possible values of A={1,1}, {2,2}, {3,3}, {4,4}, {5,5}, {6,6}.
- P(A) (Probability of A) ???????
- P(A)= \( \frac{\text{Number of ways A can occur}}{\text{number of different simple events}} \) = \( \frac{6}{36} \) = 1/6
Different ways to define Probability

• RULE 1: Relative Frequency Approximation

\[ P(A) = \frac{\text{Number of times A occurred}}{\text{Number of times a trial was repeated}} \]

Example: \( P(\text{Seeing a Mountain Lion at any day during winter}) \)

• RULE 2: Classical approach (requires equally likely outcomes). If event A can occur in \( s \) of \( n \) ways, then:

\[ P(A) = \frac{\text{number of ways A can occur}}{\text{number of different simple events}} \]

Example: \( P(\text{Getting a 4}) \) when you roll a balanced die = \( \frac{1}{6} \)
Different ways to define Probability

- **RULE 3: Subjective Probability.** $P(A)$ is estimated by using previous knowledge.
- **Example:** $P(\text{Candidate A wins an election})$

- **LAW OF LARGE NUMBERS:**
  When a procedure is repeated again and again, the relative frequency probability tends to approach the actual probability.
More on Probability

- Complement of an event:

Notation: $\bar{A}$ or $A^c$
Consists of all outcomes in which event $A$ does not occur.
Example: $P($Not getting a 5$) = P($Getting 1, 2, 3, 4 or 6$)

\[
P(\bar{A}) = 1 - P(A)
= 1 - \frac{1}{6} = \frac{5}{6}
\]
More on Probability

• A COMPOUND EVENT: Is any event combining two or more simple events

• Example: Getting an even number when rolling a die={2,4,6}

• RULES OF PROBABILITY:
For any event A

1. $0 \leq P(A) \leq 1$
2. If $P(A)=1$, A always occurs
3. If $P(A)=0$, A never occurs
4. $P(A) + P(\overline{A}) = 1 \rightarrow P(\overline{A}) = 1 - P(A)$
Union, Intersection and Disjoint events

Example: Event A= Getting a head when a coin is tossed; Event B= Getting a 5 when a single die is rolled

• UNION OF EVENTS:
P(A ∪ B) = P(A or B) = P(Event A occurs or event B occurs or they both occur) = P(Getting a head or getting a 5)

• INTERSECTION OF EVENTS:
P(A ∩ B) = P(A and B) = P(Event A occurs and event B occurs simultaneously) = P(Getting a head and getting a 5)

• DISJOINT EVENTS (or mutually exclusive):
They can no occur simultaneously
Venn Diagrams

- **UNION:**
  A or B (at least one of them occur)

- **INTERSECTION:**
  A and B (both of them occur simultaneously)
Addition Rule

• \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)

If the events are disjoint, \( P(A \cap B) = 0 \)

In this case \( P(A \cup B) = P(A) + P(B) \)

**Example:** A die is rolled. What is the probability of getting a 1 or a 6?.

\( A=\{1\}; \ B=\{6\} \). A and B are mutually exclusive.

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\]

This is zero
Independence

• **CONDITIONAL PROBABILITY NOTATION**: $P(B|A) = P(\text{event B occurs after event A has already occurred}) = P(B \text{ given A})$

• **INDEPENDENT EVENTS**

Two events A and B are independent if the occurrence of one does not affect the probability of occurrence of the other. This means $P(B|A) = P(B)$

**Example**: When tossing a coin twice, $P(\text{Head in a second toss of a coin } | \text{ Tail in a first toss of a coin}) = P(\text{Head in the second toss of a coin}) = \frac{1}{2}$
MULTIPLICATION RULE

- \( P(A \cap B) = P(A \text{ and } B) = P(A) \times P(B|A) \)

- If \( A \) and \( B \) are independents
  \( P(A \cap B) = P(A \text{ and } B) = P(A) \times P(B) \) (because \( P(B|A) = P(B) \))

Example: A coin is tossed and a die is rolled.

\[ P(\text{Getting a head and Getting a 5}) = P(\text{Head} \text{ and } 5) = \]
\[ P(A \cap B) = P(A) \times P(B|A) = P(A) \times P(B) = \]
\[ = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \]

- \( A \) and \( B \) are independent
Probability of “at least one”

- This means probability of one or more. Sometimes is easier to calculate the probability of the complement of the event.

Example: Probability that a couple has at least 1 girl among 3 children (assumes boys and girls are equally likely)
- \( A \): Getting at least 1 girl among 3 children
- \( \bar{A} \): Not getting at least 1 girl among the 3 children

\[
P(\bar{A}) = P(\text{Boy} \cap \text{Boy} \cap \text{Boy}) = P(\text{Boy}) \times P(\text{Boy}) \times P(\text{Boy})
\]
\[
= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}
\]

\[
P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{8} = \frac{7}{8}
\]
Conditional Probability. Bayes Theorem

- $P(A \text{ given } B) = P(A|B)$
- $P(B \text{ given } A) = P(B|A)$

*From the multiplication rule:*

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Therefore $P(A \cap B) = P(A|B). P(B)$
- $P(B|A) = \frac{P(A \cap B)}{P(A)}$. Therefore $P(A \cap B) = P(A). P(B|A)$

*Bayes Theorem: Dealing with sequential information*

$$P(A|B) = \frac{P(A). P(B|A)}{[P(A). P(B|A)] + [P(\bar{A}). P(B|\bar{A})]}$$
Risk and Odds

- Results of a prospective study. Risk is given as a probability value. Example: \( P(\text{Disease}) \) in the treatment group = \( \frac{a}{a+b} \)

<table>
<thead>
<tr>
<th></th>
<th>Disease</th>
<th>No Disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>Placebo</td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

Absolute Risk Reduction:
\[
|P(\text{event occurring in treatment Group}) - P(\text{event occurring in control group})| = \left| \frac{a}{a+b} - \frac{c}{c+d} \right| = |p_t - p_c|
\]
Relative Risk

- \( p_t \)= Proportion (or incidence rate) of some characteristic in the treatment group. Example: Proportion of heart attacks in the treatment group

- \( p_c \)=Proportion (or incidence rate) of some characteristic in the control group. Example: Proportion of heart attacks in the control group

Relative Risk = \( \frac{p_t}{p_c} = \frac{a}{\frac{a+b}{c}} = \frac{ac}{c+d} \) (Risk Ratio)
Odds Ratio (Prospective or Retrospective studies)

• Odds against an event A: \( \frac{P(\overline{A})}{P(A)} \), usually expressed as “m to n”, where m and n are integers with common factors. It is usually expressed as “m to n”, such as 5:1 or 5 to 1.

• Odds in favor of an event A: \( \frac{P(A)}{P(\overline{A})} \), usually expressed as “n to m”, such as 30:1 or “30 to 1”.
Odds Ratio or Relative Odds

- **Odds Ratio** = \( \frac{\text{odds in favor of event for treatment group}}{\text{odds in favor of event for control group}} \)

- **Odds Ratio** = \( \frac{ad}{bc} \)

**NOTE:**
- Use relative risk and/or odds ratio in Prospective studies
- Use odds ratio in Retrospective studies