AMS 7
Probability
Lecture 4

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• Reminder: If event $A$ is a set of $s$ simple events, then it has probability $P(A) = \frac{s}{n}$, where $n$ is the number of different simple events.

**Rules of Probability**

- $0 \leq P(A) \leq 1$
- If $P(A) = 1$, event $A$ always occurs.
- If $P(A) = 0$, event $A$ never occurs.

★★ $P(A) + P(\text{not } A) = 1$, because either $A$ or $\text{not } A$ must occur

i.e., $P(\text{not } A) = 1 - P(A)$, “not $A$” or $\bar{A}$ is called the **complement** of event $A$. 


**Examples:**

- $A = \text{rain}$  \quad $\overline{A} = \text{no rain}$
- $A = \text{roll 10}$  \quad $\overline{A} = \text{roll anything else}$

- We flip a coin 3 times. What is the probability of at least one head?

  - 1 H  \quad TTH or THT or TTH
  - 2 H  \quad THH or …
  - 3 H  \quad …

  $\rightarrow A = \text{at least one head}$

  $\rightarrow \overline{A} = \text{zero heads on three coin flips (TTT)}$

So, $P(A) = P(\overline{A}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ \quad $\Rightarrow$ \quad $P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{8} = \frac{7}{8}$. 
**Addition Rule**

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

→ The event \( A \) or \( B \) includes either \( A \) occurs or \( B \) occurs or both!!

★ **Example:** Roll at least one 1 on two die rolls:
★ If events $A$ and $B$ are **mutually exclusive** or **disjoint** only one of them can occur.

→ Mathematically:

$$P(A \text{ and } B) = 0$$

→ Graphically:

![Diagram showing mutually exclusive events $A$ and $B$]

★ **Example:** H or T

★ In this case, $P(A \text{ or } B) = P(A) + P(B)$. 
Multiplication Rule

\[ P(A \text{ and } B) = P(A) \cdot P(B|A) \]

(Prob A happens) \quad (Prob that B happens if A has already happened)

★ If events A and B are \textbf{independent} then the result of A does not affect the possibility of the result of B.

→ \textbf{Examples:} coin tosses; sampling with replacement; die rolls.

★★ Then \( P(B|A) = P(B) \Rightarrow P(A \text{ and } B) = P(A) \cdot P(B) \)

• Otherwise the events are \textbf{dependent}.
Conditional Probability:

\[ P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \]

★ Example: Suppose that probability a flight form SJC → LAX leaves on time is 0.8, and the probability a flight from JSC → LAX both leaves and arrives on time is 0.72. What is the probability such a flight arrives on time if it is known to have left on time?

\[
P(\text{arrives on time}|\text{departed on time}) = \frac{P(\text{arrives & departs on time})}{P(\text{departs on time})} = \frac{0.72}{0.8} = 0.9
\]
Bayes Theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}$$

★ Example: We have a two-headed coin and a regular coin. We pick one at random and flip it. We get a head. What is the probability it is the two-headed coin?

$$P(2H|H) = \frac{P(H|2H)P(2H)}{P(H|2H)P(2H) + P(H|1H)P(1H))}$$

$$= \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{4} + \frac{1}{4} = \frac{2}{3}$$
Relative Risk:

\[ PR = \frac{P_t}{P_c} = \frac{\text{# positive in treatment group}}{\text{total # in treatment group}} / \frac{\text{# positive in control group}}{\text{total # in control group}} \]

★★ Example:

<table>
<thead>
<tr>
<th></th>
<th>complications</th>
<th>no complications</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>treatment</td>
<td>315</td>
<td>289</td>
<td>604</td>
</tr>
<tr>
<td>control</td>
<td>304</td>
<td>293</td>
<td>597</td>
</tr>
</tbody>
</table>

\[ PR = \frac{\frac{\text{# positive in treatment group}}{\text{total # in treatment group}}}{\frac{\text{# positive in control group}}{\text{total # in control group}}} = \frac{\frac{315}{604}}{\frac{304}{597}} = 1.024 \]

- In this case the risk is the same for both groups.
Odds:

- odds in favor of event $A$: \( \frac{P(A)}{P(\overline{A})} \)

- odds ratio: \( \frac{\text{odds in favor of event for the treatment group}}{\text{odds in favor of event for the control group}} \)

Rate: describes the frequency per some reference size group. 
  e.g. per 1000.
  
- rate = relative frequency \( \times \) reference size

- Example: Infant mortality is usually given as a rate per 1000,
  
  \( \frac{\# \text{ deaths of infants in } 1^{st} \text{ year}}{\# \text{ live births}} \times 1000 \)
Key Concepts!!!!

- Complement
- Addition and Multiplication Rule
- Independence
- Conditional Probability
- Bayes Theorem