Probability measure, \( P \)

2 fair dice \( 36 \) sample points \( 36 \) mutually exclusive possibilities

\[
\frac{1}{36} \quad \text{If event } A = \{(5,6), (6,5)\} \text{ then } P(A) = \frac{2}{36}
\]

No reason to assume sample points are equally likely, we need an axiomatic definition:

Define \( F \), family of events defined on same sample space \( \Omega \)

1. \( \emptyset \) and \( \Omega \) are elements of \( F \)
2. If \( A \in F \), then \( A^c \in F \) "-" means complement
3. If \( A_1, \ldots \) are elements of \( F \), then \( \bigcup_{n=1}^{\infty} A_n \) are elements of \( F \)

Axioms of Probability Measure:

1. \( P\{A\} > 0 \) for all \( A \in F \)
2. \( P\{\emptyset\} = 1 \)
3. \( P\{A \cup B\} = P\{A\} + P\{B\} \) if events \( A \) and \( B \) are mutually exclusive (i.e., \( A \cap B = \emptyset \)) (\( A \) and \( B \) are disjoint)
4. If events \( A_1, A_2, \ldots \) are pairwise mutually exclusive (i.e., \( A_i \cap A_j = \emptyset \), \( i \neq j \))

\[
P\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P\{A_n\}.
\]

Note: \( A_1, \ldots, A_n \) pairwise mutually exclusive

\[
P\{A_1 \cup \ldots \cup A_n\} = \sum_{i=1}^{n} P\{A_i\}.
\]

Properties:

1. \( P\{\emptyset\} = 0 \). \( A \cup \emptyset = A \) \( A \cap \emptyset = \emptyset \) mutually exclusive

\[
P\{A \cup \emptyset\} = P\{A\} + P\{\emptyset\} \Rightarrow P\{\emptyset\} = 0.
\]

\[
P\{A\}
\]
2. \( P(\overline{A}) = 1 - P(A) \)  \( A \cap \overline{A} = \emptyset \)  \( A, \overline{A} \) are mutually exclusive

\( A \cup \overline{A} = \Omega \)  \( \Rightarrow P(\Omega) = P(A) + P(\overline{A}) = 1 \)

\( \Rightarrow P(A) = 1 - P(\overline{A}) \)

\[ \begin{array}{c}
\text{A, B not mutually exclusive} \\
A \cup B = (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (A \cap B)
\end{array} \]

\( P(A \cup B) = P(A \cap \overline{B}) + P(\overline{A} \cap B) + P(A \cap B) \)

\( A \cap B, A \cap \overline{B} \) are mutually exclusive \& union = \( A \)

\( P(A) = P(A \cap \overline{B}) + P(\overline{A} \cap B) \)

Similarly,

\( P(B) = P(\overline{A} \cap B) + P(A \cap \overline{B}) \)

3. \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \).

4. \( A \subset B \Rightarrow P(A) \leq P(B) \).

\( B = A \cup (B - A) \) are two disjoint (mutually exclusive) events

\( P(B) = P(A) + P(B - A) \Rightarrow P(A) \leq P(B) \geq 0 \)

Basic procedure:

1. Identify the sample space \( \Omega \)  All elements of \( \Omega \) must be mutually exclusive & collectively exhaustive.

2. Assign probabilities to elements in \( \Omega \) and those probabilities must agree with the axioms.

   * estimates based on past experience
   * analysis of random experiment
   * make assumptions, i.e., we often assume "equally likely"
3. Identify the event of interest
   statement $\leftrightarrow$ subset of $\mathbb{N}$

4. Compute probabilities of events of interest.

**Example #1**

5 devices, simultaneously active
1 bug when only 1 device is available: error
What's probability of error?

$\text{device} = \{0, 1, \text{busy}, \text{available}\}$

Event = 5 tuple of 0s and 1s

$\{0, 0, \ldots , 1, 0, 0, \ldots \}$

$\mathcal{N} = 2^5$ sample points

Assume points are equally likely.
Every point has $P(\text{point}) = \frac{1}{32}$

$\text{Event } E = \{S_1, S_2, S_4, S_8, S_{16}\}$

$P(E) = \frac{5}{32}$

**Example #2.**

100 routines
20 syntax issues
10 I/O issues
5 other issues
6 syntax & I/O issues
3 syntax & other issues
2 I/O & other issues
1 all issues

$P(\text{error}) = \frac{25}{100}$

$P(\text{syntax or I/O or both}) = P(\text{syntax I/O})$

$P(SUIV03) = P(\text{syntax I/O})$

$P(SUIV03) = P(S3) + P(EI3) - P(SUIN)$

$P(SUIV03) = P(S3) + P(EI03) - P(SUIN00)$

$P(SUIN03) = P(S3) + P(EI03) - P(SUIN00)$
Combinatorics - science of counting
finite # of sample points
drawing a sample from a population
with replacement - vs - without replacement

\[ a \ b \ c \]

- **Definition:**
  - permutation of order \( k \) when I have a selection of \( k \) elements but order matters
  - combination - when order doesn't matter

**Example:**
\[ \{x, y, z\} \]

- permutations with replacement: \( xx, xy, xz, yx, yz, yz, zy, zy, ze \)
- permutations without replacement: \( xy, xz, yx, yz, xx, zy \)
- combinations without replacement: \( xy, xz, yz \)