HOMEWORK 5 SOLUTIONS

p. 465 # 3.

\( \Phi_X(\theta) = E(e^{\theta X}) \)

\( = \sum_{x=1}^{\infty} e^{\theta x} p(x) \)

\( = \sum_{x=1}^{\infty} e^{\theta x} 2 \left( \frac{1}{3} \right)^x \)

\( = \sum_{x=1}^{\infty} e^{\theta x} 2 e^{-x \ln(3)} \)

\( = 2 \sum_{x=1}^{\infty} e^{\theta x - x \ln(3)} \)

\( = \sum_{x=1}^{\infty} e^{(\theta - \ln(3))} \text{(geometric series)} \)

\( = \frac{2e^{\theta - \ln(3)}}{1 - e^{\theta - \ln(3)}} \)

So, \( \frac{d\Phi}{d\theta} = \frac{d}{d\theta} \frac{2e^{\theta}}{3 - e^\theta} \bigg|_{\theta=0} = \frac{8e^{\theta} - 2e^{2\theta}}{(3 - e^\theta)^2} = \frac{3}{2} \).

p. 465 # 4. \( X \) is the random variable. \( f(x) = 2x, \text{ for } x \in [0, 1] \), thus we are dealing with a (continuous) density. So we will integrate. The moment-generating function is defined:

\( M_X(t) = E[e^{tX}] \)

\( = \int_0^1 e^{tx} (2x) dx \)

\( = \frac{2e^t}{t} - \frac{2e^t}{t^2} + \frac{2}{t^2} \)

p. 465 # 6. Prove \( E[X^k] = M_X^{(k)}(0). \)

\( M_X^{(k)}(t)|_{t=0} = \frac{d^k}{dt^k} \langle \sum_{x \leq x} e^{tx} p(x) \rangle \)

\( = \sum_{x \leq x} p(x) \frac{d^k}{dt^k} e^{tx} \bigg|_{t=0} \)

\( = \sum_{x \leq x} x^k p(x) \)

\( = E[X^k] \)
p. 465 # 8. The uniform distribution $f(x) = \frac{1}{b-a}$, for $x \in (b, a)$. So $f(x)$ is a continuous density. For $t \neq 0$:

$$M_X(t) = E[e^{tX}]$$

$$= \int_a^b e^{tx} \, dx$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}.$$  

For $t = 0$:


p. 474 # 8. $X + Y + Z$ are Poisson distributed with parameter $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. $X + Z$ are Poisson distributed with parameter $\lambda = \lambda_1 + \lambda_3$. Therefore,

$$P(Y = y|X + Y + Z = t) = \frac{P(Y = y, X + Z = t - y)}{P(X + Y + Z = t)}$$

$$= \frac{P(Y = y)P(X + Z = t - y)}{P(X + Y + Z = t)} \quad \text{(independent)}$$

$$= \frac{e^{-\lambda_2 \lambda_3^y} e^{-(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_3)^{t-y}}}{e^{-(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)^t}}$$

$$\frac{y!}{(t-y)!} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}\right)^y \left(\frac{\lambda_1 + \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right)^{t-y}.$$  

p. 474 # 9. $X$ = calling time of booth person; $Y$ = calling time of person ahead of Watkins. We want to find $P(X + Y \geq 12 + t|X + Y \geq t) = P(X + Y \geq 12)$ (This is from the memoryless property of the exponential distribution, p. 288)

By the remark on p. 470, $X + Y$ is Gamma distributed. What does the Gamma distribution look like?

$$\Gamma(x, \lambda) = \frac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma(r)}.$$  

Note that $\Gamma(n + 1) = n!$. Thus, the exponential distribution looks like:

$$f(t) = \lambda e^{-\lambda t}, t \geq 0$$

$$= \Gamma(1, \lambda).$$  

$X + Y$ is distributed as $\Gamma(1 + 1, \lambda)$ with $\lambda = \frac{1}{8}$. Thus,

$$P(X + Y \geq 12) = \int_{12}^{\infty} \frac{5}{8} \frac{e^{-\frac{5}{8}x} dx}{1!}$$

$$= \frac{5 e^{-\frac{5}{8} \times 12}}{2} = 0.558.$$  

p. 484 # 2. \( P(X \geq 2) = \frac{2}{5} \). By Markov’s inequality, \( \frac{2}{5} = P(X \geq 2) \leq \frac{E[X]}{2} \). Thus, \( E[X] \geq \frac{4}{5} \).

p. 484 # 3. a. \( P(X \geq 11) = \frac{E[X]}{11} = \frac{5}{11} = 0.4545 \).

b. \( P(|X - E[X]| \geq t) \leq \frac{\sigma^2}{t^2} \) is Chebyshev’s inequality. \( P(X \geq 11) = P(|X - 5| \geq 6) \), since \( E[X] = 5 \). \( E[X^2] = 42 \), which implies \( \text{Var}[X] = -25 + 42 = \sigma^2 \). Thus, \( P(X \geq 11) \leq \frac{42 - 25}{36} = 0.472 \).

p. 484 # 4. \( X \) = the lifetime of the lightbulb. Thus, \( P(X \leq 700) \leq P(|X - 800| \geq 100) \leq \frac{\sigma^2}{10^4} = \frac{50}{10000} \), because \( \sigma^2 = E[X^2] - (E[X])^2 = 50 \).

p. 485 # 12. We are given that \( E[X] = \mu \), sample size = \( N \), and the error of estimation = \( |E[X] - \mu| \). We are also given that the probability of the error of estimation being less than \( 2\sigma \) is greater than 0.98. We write this as: \( P(|E[X] - \mu| < 2\sigma) \geq 0.98 \). Look at equation 11.3 on p.481: \( P(|E[X] - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2 N} \). Thus, 

\[
\begin{align*}
(28) & \quad 1 - P(|E[X] - \mu| < 2\sigma) < 1 - 0.98 \\
(29) & \quad P(|E[X] - \mu| \geq 2\sigma) < 0.02
\end{align*}
\]

\( \epsilon = 2\sigma \), thus \( 0.02 \geq \frac{\sigma^2}{\epsilon^2 N} \). Thus, \( N \geq \frac{1}{4(0.02)^2} \).

p. 485 # 14. Equation 11.5 implies \( n \geq \frac{\mu(1-\mu)}{\epsilon^2 \alpha} = \frac{1}{4\epsilon^2 \alpha} \). \( \alpha = 1 - 0.96 \) and \( \epsilon = 0.05 \), so \( n \geq 1666.67 \).