Section 7.1 Uniform random variables

7. Let \( X \) be a random number from \((0, \ell)\). The probability of the desired event is

\[
P\left(\min(X, \ell - X) \geq \frac{\ell}{3}\right) = P\left(X \geq \frac{\ell}{3}, \ell - X \geq \frac{\ell}{3}\right) = P\left(\frac{\ell}{3} \leq X \leq \frac{2\ell}{3}\right) = \frac{2\ell - \frac{\ell}{3}}{\ell} = \frac{1}{3}.
\]

10. Let \( F \) be the probability distribution function and \( f \) be the probability density function of \( X \). By definition,

\[
F(x) = P(X \leq x) = P(\tan \theta \leq x) = P(\theta \leq \arctan x) = \frac{\arctan x - \left(-\frac{\pi}{2}\right)}{\pi} = \frac{\arctan x + \frac{1}{2}}{\pi}, \quad -\infty < x < \infty.
\]

Thus

\[
f(x) = F'(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.
\]

12. (a) Let \( G \) and \( g \) be the distribution and density functions of \( Y \), respectively. Since \( Y \geq 0 \),

\[
G(x) = P(Y \leq x) = P(-\ln(1 - X) \leq x) = P(X \leq 1 - e^{-x}) = \frac{(1 - e^{-x}) - 0}{1 - 0} = 1 - e^{-x}.
\]
Thus
\[ g(x) = G'(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \]

(b) Let \( H \) and \( h \) be the probability distribution and probability density functions of \( Z \), respectively. For \( n > 0 \), \( H(x) = P(Z \leq x) = 0, x < 0; \)
\[ H(x) = P(Z \leq x) = P(X \leq \sqrt[n]{x}) = \frac{n}{\sqrt[n]{x}}, \quad 0 < x < 1; \]
\( H(x) = 1, \) if \( x \geq 1. \) Therefore,
\[ h(x) = H'(x) = \begin{cases} \frac{1}{n} \cdot \frac{1}{\sqrt[n]{x}^{n-1}} & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases} \]

For \( n < 0, \) \( H(x) = P(X^n \leq x) = 0, x < 1; \)
\[ H(x) = P(X^n \leq x) = P\left(X^{-n} \geq \frac{1}{x}\right) = P\left(X \geq \left(\frac{1}{x}\right)^{-\frac{1}{n}}\right) = P(X \geq x^{1/n}) = 1 - x^{1/n}, \quad x \geq 1. \]
Therefore,
\[ h(x) = \begin{cases} \frac{1}{n} \cdot \frac{1}{x^{n-1}} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases} \]

Section 7.2 Normal random variables

6. (a) \( P(X > 35.5) = P\left(\frac{X - 35.5}{4.8} > \frac{35.5 - 35.5}{4.8}\right) = 1 - \Phi(0) = 0.5. \)

(b) The desired probability is given by
\[ P(30 < X < 40) = P\left(\frac{30 - 35.5}{4.8} < X < \frac{40 - 35.5}{4.8}\right) = \Phi(0.94) - \Phi(-1.15) = \Phi(0.94) + \Phi(1.15) - 1 = 0.8264 + 0.8749 - 1 = 0.701. \]
25. For $t \geq 0$,

$$P(Y \leq t) = P(e^X \leq t) = P(X \leq \ln t) = P\left(Z \leq \frac{\ln t - \mu}{\sigma}\right) = \Phi\left(\frac{\ln t - \mu}{\sigma}\right).$$

Let $f$ be the probability density function of $Y$. We have

$$f(t) = \frac{d}{dt} P(Y \leq t) = \frac{1}{\sigma t} \Phi'\left(\frac{\ln t - \mu}{\sigma}\right), \quad t \geq 0.$$

So

$$f(t) = \begin{cases} \frac{1}{\sigma t \sqrt{2\pi}} \exp\left[-\frac{(\ln t - \mu)^2}{2\sigma^2}\right] & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

26. Let $f$ be the probability density function of $Y$. Since for $t \geq 0$,

$$P(Y \leq t) = P\left(\sqrt{|X|} \leq t\right) = P\left(|X| \leq t^2\right) = P\left(-t^2 \leq X \leq t^2\right) = 2\Phi(t^2) - 1,$$

we have that

$$f(t) = \frac{d}{dt} P(Y \leq t) = \begin{cases} 4t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Section 7.3 Exponential random variables

3. For $-\infty < y < \infty$,

$$P(Y \leq y) = P(-\ln X \leq y) = P\left(X \geq e^{-y}\right) = e^{-e^{-y}}.$$

Thus $g(y)$, the probability density function of $Y$ is given by

$$g(y) = \frac{d}{dy} P(Y \leq y) = e^{-y} \cdot e^{-e^{-y}} = e^{-y} - e^{-y}.$$

10. Mr. Jones makes his phone calls when either A or B is finished his call. At that time the remaining phone call of A or B, whichever is not finished, and the duration of the call of Mr. Jones both have the same distribution due to the memoryless property of the exponential distribution. Hence, by symmetry, the probability that Mr. Jones finishes his call sooner than the other one is 1/2.
Section 12.3 Markov chains

2. For \( j \geq 0 \),

\[
P(X_n = j) = \sum_{i=0}^{\infty} P(X_n = j \mid X_0 = i) P(X_0 = i) = \sum_{i=0}^{\infty} p_{ij}^n p(i),
\]

where \( p_{ij}^n \) is the \( ij \)th entry of the matrix \( P^n \).

12. For \( n \geq 1 \), let

\[
X_n = \begin{cases} 
1 & \text{if the } n\text{th fish caught is trout} \\
0 & \text{if the } n\text{th fish caught is not trout.}
\end{cases}
\]

Then \( \{X_n : n = 1, 2, \ldots \} \) is a Markov chain with state space \( \{0, 1\} \) and transition probability matrix

\[
\begin{pmatrix}
10/11 & 1/11 \\
8/9 & 1/9 
\end{pmatrix}
\]

Let \( \pi_0 \) be the fraction of fish in the lake that are not trout, and \( \pi_1 \) be the fraction of fish in the lake that are trout. Then, by Theorem 12.7, \( \pi_0 \) and \( \pi_1 \) satisfy

\[
\begin{pmatrix}
\pi_0 \\
\pi_1
\end{pmatrix} = \begin{pmatrix}
10/11 & 8/9 \\
1/11 & 1/9
\end{pmatrix} \begin{pmatrix}
\pi_0 \\
\pi_1
\end{pmatrix},
\]

which gives us the following system of equations

\[
\begin{align*}
\pi_0 &= (10/11)\pi_0 + (8/9)\pi_1 \\
\pi_1 &= (1/11)\pi_0 + (1/9)\pi_1.
\end{align*}
\]

By choosing any one of these equations along with the relation \( \pi_0 + \pi_1 = 1 \), we obtain a system of two equations in two unknowns. Solving that system yields \( \pi_0 = 88/97 \approx 0.907 \) and \( \pi_1 = 9/97 \approx 0.093 \). Therefore, approximately 9.3% of the fish in the lake are trout.
Section 12.4 Continuous time Markov chains

2. Clearly, \( \{X(t) : t \geq 0\} \) is a continuous-time Markov chain. Its balance equations are as follows:

<table>
<thead>
<tr>
<th>State</th>
<th>Input rate to ( \mu \pi_0, \mu \pi_1, \mu \pi_2, \mu \pi_3 )</th>
<th>Output rate from ( \lambda \pi_f, \lambda \pi_1, \lambda \pi_2, \mu \pi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>( \mu \pi_0 = \lambda \pi_f )</td>
<td>( \lambda \pi_0 = \lambda \pi_1 + \mu \pi_1 )</td>
</tr>
<tr>
<td>0</td>
<td>( \lambda \pi_f + \mu \pi_1 + \mu \pi_2 + \mu \pi_3 = \mu \pi_0 + \lambda \pi_0 )</td>
<td>( \lambda \pi_0 = \lambda \pi_1 + \mu \pi_1 )</td>
</tr>
<tr>
<td>1</td>
<td>( \lambda \pi_0 = \lambda \pi_1 + \mu \pi_1 )</td>
<td>( \lambda \pi_1 = \lambda \pi_2 + \mu \pi_2 )</td>
</tr>
<tr>
<td>2</td>
<td>( \lambda \pi_1 = \lambda \pi_2 + \mu \pi_2 )</td>
<td>( \lambda \pi_2 = \mu \pi_3 )</td>
</tr>
</tbody>
</table>

Solving these equations along with

\[ \pi_f + \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \]

we obtain

\[ \pi_f = \frac{\mu^2}{\lambda (\lambda + \mu)}, \quad \pi_0 = \frac{\mu}{\lambda + \mu}, \]
\[ \pi_1 = \frac{\lambda \mu}{(\lambda + \mu)^2}, \quad \pi_2 = \frac{\lambda^2 \mu}{(\lambda + \mu)^3}, \]
\[ \pi_3 = \left( \frac{\lambda}{\lambda + \mu} \right)^3. \]

4. Let \( X(t) \) be the number of customers in the system at time \( t \). Then the process \( \{X(t) : t \geq 0\} \) is a birth and death process with \( \lambda_n = \lambda, \ n \geq 0, \) and \( \mu_n = n \mu, \ n \geq 1. \) To find \( \pi_0 \), the probability that the system is empty, we will first calculate the sum in (12.18). We have

\[
\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu \mu_2 \cdots \mu_n} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n! \mu^n} = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n = -1 + \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n = -1 + e^{\lambda/\mu}.
\]

Hence, by (12.18),

\[ \pi_0 = \frac{1}{1 - 1 + e^{\lambda/\mu}} = e^{-\lambda/\mu}. \]
By (12.17),
\[ \pi_n = \frac{\lambda^n \pi_0}{n! \mu^n} = \frac{(\lambda / \mu)^n e^{-\lambda / \mu}}{n!}, \quad n = 0, 1, 2, \ldots \]

This shows that the long-run number of customers in such an $M/M/\infty$ queueing system is Poisson with parameter $\lambda / \mu$. The average number of customers in the system is, therefore, $\lambda / \mu$.

10. Let $X(t)$ be the population size at time $t$. Then $\{X(t) : t \geq 0\}$ is a birth and death process with birth rates $\lambda_n = n\lambda + \gamma, \ n \geq 0$, and death rates $\mu_n = n\mu, \ n \geq 1$. For $i \geq 0$, let $H_i$ be the time, starting from $i$, until the population size reaches $i + 1$ for the first time. We are interested in $E(H_0) + E(H_1) + E(H_2)$. Note that, by Lemma 12.2,
\[ E(H_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E(H_{i-1}), \quad i \geq 1. \]

Since $E(H_0) = 1 / \gamma$,
\[ E(H_1) = \frac{1}{\lambda + \gamma} + \frac{\mu}{\lambda + \gamma} \cdot \frac{1}{\gamma} = \frac{\mu + \gamma}{\gamma(\lambda + \gamma)}, \]

and
\[ E(H_2) = \frac{1}{2\lambda + \gamma} + \frac{2\mu}{2\lambda + \gamma} \cdot \frac{\mu + \gamma}{\gamma(\lambda + \gamma)} = \frac{\gamma(\lambda + \gamma) + 2\mu(\mu + \gamma)}{\gamma(\lambda + \gamma)(2\lambda + \gamma)}. \]

Thus the desired quantity is
\[ E(H_0) + E(H_1) + E(H_2) = \frac{(\lambda + \gamma)(2\lambda + \gamma) + (\mu + \gamma)(2\lambda + 2\mu + \gamma) + \gamma(\lambda + \gamma)}{\gamma(\lambda + \gamma)(2\lambda + \gamma)}. \]