HW # 2

Series system $A \rightarrow \text{components} \rightarrow B$

The whole system fails if any component fails.

Parallel system $A \rightarrow \text{components} \rightarrow B$

Assume failures are mutually independent.

Analyze series system $n$ components

$A_i$: component $i$ is "OK"

Reliability of component $i = R_i = P\{A_i\} \text{ is "OK"}$

$P\{A_1 \land A_2 \land \ldots \land A_n\} = \prod_{i=1}^{n} R_i$, since they are independent

Product law of reliability for series system

As complexity increases, reliability decreases.

Imagine you have identical components and $R_i = 0.98$

Then, 5 components $R_{\text{total}} = 0.904$

$10$ components $R_{\text{total}} = 0.817$
Look at system with fully parallel design

$A_p$ means parallel system of $n$ components is OK

$$R_p = P\{A_p\}$$

Look at complement, $\overline{A_p}$, when system fails:
$$P\{\overline{A_p}\} = P\{\overline{A_1 \cap A_2 \cap \ldots \cap A_n}\}$$

$$F_p = 1 - R_p \Rightarrow P\{\overline{A_1}\} P\{\overline{A_2}\} \ldots P\{\overline{A_n}\}$$

Unreliability of system

$$R_p = 1 - F_p = 1 - \prod_{i=1}^{n} P\{1 - R_i\} \quad \text{suffers from law of diminishing returns}$$

Imagine series design but each stage is $n_i$ replicated,

$n$ serial stages

$A \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_4 \rightarrow C_5 \rightarrow B$

Component $i$ at stage $i$ has $R_i$ reliability

$A_i$ component $i$ is OK

$$R_i = P\{A_i\} \quad i=1,2,\ldots,5$$

We want overall system reliability

$$R = P\{A\} \quad \text{getting from } A \text{ to } B$$

$$A = (A_1 \cap A_4) \cup (A_2 \cap A_4) \cup (A_2 \cap A_5) \cup (A_3 \cap A_5)$$

But these events are not mutually exclusive
If $C_2$ OK then $C_1, C_2$ - who cares
$C_2$ fails then 2 parallel branches

\[ P_{EA_3} = P_{EA_2 | A_1} R_2 + P_{EA_2 | \bar{A}_1} \bar{A}_2 \frac{1-R_2}{R_2} \]

Using law of total probability

\[ P_{EA | A_2} = P_{EA | A_4 \cup A_5} = 1 - (1-R_4)(1-R_5) \]

because components fail independently

\[ P_{EA | \bar{A}_2} = 1 - (1-R_1R_4)(1-R_3R_5). \]

\[ R_R = \text{not OK} \]
\[ R_s \text{system} = R_2 \left\{ 1 - (1-R_4)(1-R_5) \right\} \]
\[ + (1-R_2) \left\{ 1 - (1-R_1R_4)(1-R_3R_5) \right\}. \]

Consider binary channel - codes 0 & 1 and is noisy.
0 \rightarrow 0 0.94
1 \rightarrow 1 0.91

\[ P\{0\} = 0.45 = \text{probability of sending "0"} \]

1. Determine prob. that 1 is received

\[ P\{R_0\} = 1. \]

2. Determine prob. that 0 is received

\[ P\{R_0\} = 1. \]

3. Determine prob. that 1 was sent given that 0 was received

\[ P\{1 | R_0\} = 1. \]

4. Determine prob. of an error

\[ T_0 = \text{event that 0 was sent} \]
\[ R_0 = \text{received} \]
\[ T_1 = \text{received} \]
\[ R_1 = \text{received} \]
5. Prob of error $P\{ (T_1 \cap R_0) \cup (T_0 \cap R_1) \}$

\[ P \{ R_0 \mid T_0 \} = 0.94 \]
\[ P \{ R_1 \mid T_1 \} = 0.91 \]
\[ P \{ T_0 \} = 0.45 \]

\[ P \{ R_1 \mid T_0 \} = 1 - 0.94 \]
\[ P \{ R_0 \mid T_1 \} = 1 - 0.91 \]
\[ P \{ T_1 \} = 0.55 \]

\[ P \{ R_0 \} = P \{ R_0 \mid T_1 \} P \{ T_1 \} + P \{ R_0 \mid T_0 \} P \{ T_0 \} \text{ by the law of total prob.} \]
\[ = 0.91 \times 0.55 + 0.94 \times 0.45 = 0.4725 \]
\[ P \{ R_1 \} = 1 - P \{ R_0 \} = 0.5275 \]

\[ P \{ T_1 \mid R_0 \} = \frac{P \{ R_1 \mid T_1 \} P \{ T_1 \}}{P \{ R_0 \}} \text{ by Bayes rule} \]
\[ = \frac{0.91 \times 0.55}{0.5275} \]

\[ P \{ T_0 \mid R_0 \} = \frac{P \{ R_0 \mid T_0 \} P \{ T_0 \}}{P \{ R_0 \}} \]

\[ P \{ \text{error} \} = P \{ T_1 \cap R_0 \} + P \{ T_0 \cap R_1 \} \]
\[ = P \{ T_1 \} P \{ R_0 \mid T_1 \} + P \{ T_0 \} P \{ R_1 \mid T_0 \} \]
\[ \approx 7.6\% \]
Random variables associates a number with the outcome

$X$ is a real-valued function defined on sample space $\Omega$

$X : \Omega \to \mathbb{R}$

$x = x$ abbreviation of $\{ \omega : \omega \in \Omega \text{ and } X(\omega) = x \}$

$x \leq x$ abbreviation of $\{ \omega : \omega \in \Omega \text{ and } X(\omega) \leq x \}$

$y < X \leq x$ abbreviation of $\{ \omega : \omega \in \Omega \text{ and } y < X(\omega) \leq x \}$

cumulative distribution function = CDF = $F$

$F(x) = P(\{ X \leq x \}) \text{ for all } x \in \mathbb{R}$

Example.

$F(6) = P(\{ X \leq 6 \})$ sum of two dice faces

$F$ non-decreasing: if $x < y$ then $F(x) \leq F(y)$

$$\lim_{x \to \infty} F(x) = 1.$$  

$$\lim_{x \to -\infty} F(x) = 0.$$  

$P(\{ x < X \leq y \}$ semi-open interval $x \quad y$

$\{ x < X \leq y \}$ and $\{ X \leq x \}$ are mutually exclusive

$\{ X \leq y \} = \{ x < X \leq y \} \cup \{ X \leq x \}$

$P(\{ X \leq y \} = P(\{ x < X \leq y \} + P(\{ X \leq x \}$
\[ F(y) = \Pr\{x < X \leq y\} + F(x) \]
\[ \Pr\{x < X \leq y\} = F(y) - F(x) \]

\[ p(x) = \Pr\{X = x\} \quad \text{only makes sense for discrete R.V.'s is called "p.m.f." (Probability Mass Function)} \]

"mass points" — pts at which there is nonzero probability

\[ p(x) \geq 0 \quad \text{for every } x \in D \text{ (domain of } X) \]

\[ \sum_{\text{all masspoints}} \Pr(x) = 1 \]

Ex. dice — sum of two faces
\[ X = 2, 3, \ldots, 12 \]
\[ \Pr\{X = 2\} = \frac{1}{36} \quad \Pr\{X = 3\} = \frac{2}{36} \quad \cdots \quad \Pr\{X = 12\} = \frac{1}{36} \]
\[ \Pr\{X = 4\} = p(4) = \frac{3}{36} \]
\[ F(3) = p(2) + p(3). \]
\[ : \]
\[ F(12) = 1. \]

For continuous R.V.s → Probability Density Function
\[ f(x) \quad x \in \mathbb{R} \quad \text{s.t.} \quad f(x) \geq 0 \]
\[ \int_a^b f(x) \, dx = \Pr\{a < X \leq b\} = \Pr\{a < X < b\} \]
\[ \int_{-\infty}^{+\infty} f(x) \, dx = 1 \]

\[ \Pr\{X = c\} = 0 \quad \forall c \in \mathbb{R} \quad \text{if } X \text{ is continuous random variable} \]

CDF \[ F(x) = \int_{-\infty}^{x} f(t) \, dt \]