Markov process 
\( \{X(t), t \in T \} \)

If 
\[ P \{ X(t_{n+1}) = x_{n+1} \mid X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = x_n \} \]

\[ = P \{ X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n \} \]

then the stochastic process is a Markov process.

If state space is discrete then the stochastic process is a Markov chain with parameter discrete or continuous time \( t \)

If \( t \) is discrete \( t_n, n=1, 2, \ldots \)

\( \{X_n\} \sim X(t_n) \)

\( X_0 = i \xrightarrow{} X_1 = j \xrightarrow{} \) like a frog in a pond with lilypads

jumps = state transition

\[ P \{ X_{n+1} = j \mid X_n = i \} \]

could depend on \( n \)

transition probabilities

\( p_{ij} \)

MC homogeneous in time

\[ \sum_j p_{ij} = 1. \]
Market in favor out of favor - no longer making money

50% will remain successful through end of next week
2/5 probability of another hit
3/5 probability am of remaining unprofitable

\[ \frac{1}{2} \quad 1 \quad 2 \quad 2 \quad 3/5 \]
Markov chain transition diagram

\[ p_{11} = 0.5 \quad p_{12} = 0.5 \]
\[ p_{21} = \frac{2}{5} \quad p_{22} = \frac{3}{5} \]

What is probability that OK after n weeks after we start?

\[ \Pi_i^{(n)} \text{ will be in state } i \text{ after n transitions} \]
\[ \Pi_j^{(n+1)} = \sum_{i} \Pi_i^{(n)} p_{ij} \quad n = 0, 1, \ldots \]

\( \Pi^{(n)} \) row vector of \( \Pi_i^{(n)} \)

\[ \Pi^{(n+1)} = \Pi^{(n)} P^{n} \]

\( \Pi^{(n)} = \Pi^{(0)} P^{n} \)

\[ \Pi^{(0)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \]
\[ \Pi^{(1)} = \Pi^{(0)} P^{1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \]

\[ \Pi^{(2)} = \Pi^{(1)} P^{1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \]

<table>
<thead>
<tr>
<th>state</th>
<th>( p_i^{n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5 0.45 0.445 0.4445 0.4/9</td>
</tr>
<tr>
<td>2</td>
<td>0.5 0.55 0.555 0.5555 0.5/9</td>
</tr>
</tbody>
</table>
Start now w/ unsuccessful toy

\[
\begin{bmatrix}
0 & 1 \\
\frac{2}{5} & \frac{3}{5}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{2}{5} & \frac{3}{5}
\end{bmatrix}
= \begin{bmatrix}
\frac{2}{5} & \frac{3}{5}
\end{bmatrix}
= \begin{bmatrix}
0.4 & 0.6
\end{bmatrix}
\]

\[\begin{array}{cccc}
n & 0 & 1 & 2 & 3 & 4 & \text{Long-term} \\
\text{successful} & 0 & 0.4 & 0.44 & 0.444 & 0.4444 & 4/9 \\
\text{not successful} & 1 & 0.6 & 0.56 & 0.556 & 0.5556 & 5/9 \end{array}\]

It does not matter how you start.

\[\lim_{n \to \infty} \pi_j^{(n+1)} = \sum_i \pi_i^{(n)} p_{ij}\]

\[\pi_j = \sum_i \pi_i p_{ij}\]

1) \(p_1 = \frac{1}{2} p_1 + \frac{2}{5} p_2\)

2) \(p_2 = \frac{1}{2} p_1 + \frac{3}{5} p_2\)

3) \(p_1 + p_2 = 1\)

3 eqns 2 variables so one eqn is redundant.

\[\frac{1}{2} p_1 = \frac{2}{5} p_2 \Rightarrow p_1 = \frac{4}{9} \quad p_2 = \frac{5}{9}\]

state \(j\) is reachable from \(i\)

Every state is reachable \(\Leftrightarrow\) Markov chain is irreducible.

\[P\{X_n = j \mid X_0 = i\} \overset{def}{=} \pi_j^{(n)}\]

State is periodic with period \(d\) if \(d \geq 1\)

\[P_{ii}^{(n)} > 0 \quad n = d, 2d, \ldots\]

If \(d = 1\), state is aperiodic: \(P_{ii}^{(n)} > 0\), for all \(n=1,2,\ldots\).
Given state $i$

$$f_i \quad \text{Prob 1st return to state } i \quad \text{occurs after } n \text{ transitions}$$

$$\mathbb{E}_i \text{ of never returning to state } i$$

$$f_i = \sum_{n=1}^{\infty} f_i^{(n)}$$

Case 1. $f_i < 1$ $i$ is said to be transient

not sure you will return

Case 2. $f_i = 1$ $i$ is "recurrent" - recurs w/ prob 1.

Mean recurrence time

$$m_i = \sum_{n=1}^{\infty} f_i^{(n)} \cdot n$$

Either $m_i = \infty \Rightarrow "\text{recurrent null}"$

$$m_i < \infty \Rightarrow "\text{positive recurrent}"$$

Reachable $\Rightarrow$ irreducible

Periodic/aperiodic

Transient/recurrent null/pos recurrent

Types of states

Theorem: If $\{X_n\}$ is an irreducible Markov chain

only one statement holds, i.e. all states are pos. rec.

all states are rec. null

$$\pi_i^{(n)} \text{ - after } n \text{ transition state is } j$$

$$j = P \{X_n = j \}$$

$$\pi_j^{(0)} = P \{X_0 = j \}$$
Markov chain has stationary distribution if state vector $\pi = \pi P$

Compare to:

**Long-run or limiting vector:**

$$\lim_{n \to \infty} \pi_j^{(n)} = \lim_{n \to \infty} P \{ X_n = j \}$$

**IF**

$\{X_n\}$ irreducible

aperiodic

time-homogeneous

Then $\lim_{n \to \infty} \pi_j^{(n)}$ always exists AND

there are 2 cases

**1. All**

states are transient or recurrent null

$$\Rightarrow \pi_j = 0 \ \forall j$$

**2. All**

states are positive recurrent

then $\{X_n\}$ is ergodic

then $\pi_j > 0 \ \forall j$ AND $\pi_j = \frac{1}{m_j}$

**IF** finite state Markov chain $\{X_n\}$ irreducible, aperiodic $\Rightarrow$ ergodic

- why all finite queuing systems are always stable.

**IF** Markov chain $\{X_n\}$ irreducible, aperiodic

$$\Leftrightarrow \text{ergodic (there exists a steady state)}$$

non-null solution of $\sum_j X_j p_{ji} = x_i \ \forall i$
s.t. $\sum_{i} |x_i| < \infty$

Recall $\sum p_i = 1$. 