Probability Rules with Outside Conditioning

The basic probability rules hold when everything is conditioned on some outside event. This short document proves the basic identities in this situation. Recall the chain rule for probabilities (the product rule)


the definition of conditional probability,

\[ P[A \mid B] = \frac{P[A, B]}{P[B]} \]

the sum rule for probabilities (assuming the \( X_i \) are disjoint events partitioning the outcome space)

\[ P[A] = \sum_i P[A, X_i] = \sum_i P[X_i] P[A \mid X_i] \]

and Bayes’ rule:

\[ P[A \mid B] = \frac{P[B \mid A] P[A]}{P[B]} . \]

Derivation of conditional chain rule:


Derivation of definition of conditional probability under conditioning:

\[ P[A \mid B, S] = \frac{P[A, B, S]}{P[B, S]} = \frac{P[A, B \mid S] P[S]}{P[B \mid S] P[S]} = P[A, B \mid S] / P[B \mid S] \]

Derivation of conditional version of sum rule:

\[ P[A \mid S] = \frac{P[A, S]}{P[S]} = \frac{\sum_i P[A, X_i, S]}{P[S]} = \sum_i \frac{P[A, X_i, S]}{P[S]} = \sum_i P[A, X_i \mid S] = \sum_i P[A \mid X_i, S] P[X_i \mid S] \]

Derivation of conditional version of Bayes’ rule:

\[ P[A, B \mid S] = P[B, A \mid S] \]

\[ P[A \mid B, S] P[B \mid S] = P[B \mid A, S] P[A \mid S] \]

\[ P[A \mid B, S] = P[B \mid A, S] P[A \mid S] / P[B \mid S] \]

Note that some of the conditional derivations use previous conditional probability facts, and commas indicate the “and” of the listed events.